KNOT SURGERY FORMULAE FOR INSTANTON FLOER HOMOLOGY I: THE MAIN THEOREM

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Abstract. We prove an integral surgery formula for framed instanton homology $I^\sharp(Y_m(K))$ for any knot K in a 3-manifold Y with $[K] = 0 \in H_1(Y;\mathbb{Q})$ and $m \neq 0$. Though the statement is similar to Ozsváth-Szabó's integral surgery formula for Heegaard Floer homology, the proof is new and based on sutured instanton homology SHI and the octahedral lemma in the derived category. As a corollary, we obtain an exact triangle between $I^\sharp(Y_m(K))$, $I^\sharp(Y_{m+k}(K))$ and k copies of $I^\sharp(Y)$ for any $m \neq 0$ and large k. In the proof of the formula, we discover many new exact triangles for sutured instanton homology and relate some surgery cobordism map to the sum of bypass maps, which are of independent interest. In a companion paper, we derive many applications and computations based on the integral surgery formula.

Contents

1. Introduction

Framed instanton homology $I^\sharp(Y)$ for any closed 3-manifold Y was introduced by Kronheimer-Mrowka [KM11] and was conjectured to be isomorphic to the hat version of Heegaard Floer homology $\widehat{HF}(Y)$ [KM10]. This conjecture is still widely open and, due to the computational difficulty of instanton Floer homology, not many examples are known. In recent years, many people have done computations of the framed instanton homology special families of 3-manifolds, see for example [LPCS20, BS19, BS21]. Yet most results focused on computing the dimension of framed instanton Floer homology and many techniques only work for S^3 or rational homology spheres, but a general structural theorem that relates the framed instanton homology of Dehn surgeries to the information from the knot complement still remains elusive.

In [LY21b], the authors of the current paper proved a large surgery formula for framed instanton homology which led to a series of applications in computing the framed instanton homology and studying the representations of the fundamental groups of Dehn surgeries of some families of knots. However, in that work, the Dehn surgery slope must be large (at least $2g+1$ where g is the Seifert genus of the knot), and thus still not much is known about the framed instanton homology of small Dehn surgery slopes. In this paper, we further prove an integral surgery formula for rationally null-homologous knots, inspired by Ozsváth-Szabó's surgery formula for Heegaard Floer homology [OS08, OS11]. For simplicity, in the introduction we only present the discussions and results for (integral) null-homologous knots (*e.g.* knots in S^3) and leave the general setups to Section 3.3.

First let us recall the results from [LY21b]. Suppose $K\subset Y$ is a null-homologous knot. Let $Y\backslash N(K)$ be the knot complement and let Γ_μ be the union of two oppositely oriented meridians of the knot on $\partial(Y\backslash N(K))$. Let $SHI(-Y\backslash N(K),-\Gamma_\mu)$ be the corresponding sutured instanton homology introduced by Kronheimer-Mrowka [KM10], where the minus sign denotes the orientation reversal for technical needs (note that $SHI(-M,-\gamma)\cong SHI(M,\gamma)$ and in particular $I^\sharp(-Y_{-m}(K))\cong I^\sharp(Y_{-m}(K))$). A Seifert surface of K induces a $\mathbb{Z}$-grading on $SHI(-Y\backslash N(K),-\Gamma_\mu)$. In [LY21b], we constructed a set of differentials on $SHI(-Y\backslash N(K),-\Gamma_\mu)$

$$d^i_j: SHI(-Y\backslash N(K),-\Gamma_\mu,i)\to SHI(-Y\backslash N(K),-\Gamma_\mu,j)$$

for any gradings $i\neq j\in\mathbb{Z}$. We then constructed bent complexes

$$A_s=\left(SHI(-Y\backslash N(K),-\Gamma_\mu),\sum_{s\leqslant i<j}d^i_j+\sum_{s\geqslant i>j}d^i_j\right),$$

$$B^+=\left(SHI(-Y\backslash N(K),-\Gamma_\mu),\sum_{i<j}d^i_j\right),\text{ and } B^-=\left(SHI(-Y\backslash N(K),-\Gamma_\mu),\sum_{i>j}d^i_j\right).$$

From [LY21b], the homologies of these complexes are related to the Dehn surgeries of K as follows:

$$H(B^+)\cong H(B^-)\cong I^\sharp(-Y), \tag{1.1}$$

$$I^\sharp(Y_{-m}(K))\cong\bigoplus_{s=\lfloor\frac{1-m}{2}\rfloor}^{\lfloor\frac{m-1}{2}\rfloor}H(A_s)\text{ for any integer } m\geqslant 2g(K)+1. \tag{1.2}$$

To state the integral surgery formula, we introduce more notations. For $s\in\mathbb{Z}$, let $B^\pm_s$ be identical copies of $B^\pm$. Define chain maps

$$\pi^{\pm,s}:A_s\to B^\pm_s$$

as follows: for $x \in (SHI(-Y\backslash N(K), -\Gamma_\mu), i)$,

$$\pi^{+,s}(x) = \begin{cases} x & i \geqslant s, \\ 0 & i < s, \end{cases} \text{ and } \pi^{-,s}(x) = \begin{cases} x & i \leqslant s, \\ 0 & i > s. \end{cases}$$

Let $\pi^{\pm}$ denote the direct sum of all $\pi^{\pm,s}$. Slightly abusing the notation, we also use them to denote the induced maps on the homologies. The main result of the paper is the following.

Theorem 1.1 (Integral surgery formula)**.** *Suppose $K \subset Y$ is a null-homologous knot. Let A_s, $B_s^{\pm}$, $\pi^{\pm}$ be defined as above. Then for any $m \in \mathbb{Z}\backslash\{0\}$, there exists a grading preserving isomorphism*

$$\Xi_m : \bigoplus_{s\in\mathbb{Z}} H(B_s^+) \xrightarrow{\cong} \bigoplus_{s\in\mathbb{Z}} H(B_{s+m}^-)$$

so that

$$I^\sharp(-Y_{-m}(K)) \cong H\left(\operatorname{Cone}(\pi^- + \Xi_m \circ \pi^+ : \bigoplus_{s\in\mathbb{Z}} H(A_s) \to \bigoplus_{s\in\mathbb{Z}} H(B_s^-))\right).$$

With the isomorphisms in (1.2) and (1.1), we can truncate the above formula for $I^\sharp(Y_{-m}(K))$ to obtain the following exact triangle.

Corollary 1.2 (Generalized surgery exact triangle)**.** *Suppose $K \subset Y$ is a null-homologous knot and m is a fixed nonzero integer. Then for any large enough integer k, there exists an exact triangle*

$$\begin{tikzcd}[column sep=small] I^\sharp(-Y_{-m-k}(K)) \arrow[rr] & & \bigoplus_{i=1}^k I^\sharp(-Y) \arrow[dl] \\ & I^\sharp(-Y_{-m}(K)) \arrow[ul] & \end{tikzcd} \tag{1.3}$$

Remark 1.3. Note that in Theorem 1.1 we exclude the case of $m = 0$. This is due to the sign ambiguity in the definition of sutured instanton homology. The original version of sutured instanton homology defined by Kronheimer-Mrowka [KM10] was only well-defined up to isomorphisms, and then Baldwin-Sivek [BS15] proved that they are well-defined up to a scalar in $\mathbb{C}$. As a result, all related maps are only well-defined up to scalars. When $m \neq 0$, the maps $\pi^{+,s}$ and $\Xi_m \circ \pi^{-,s}$ have distinct targeting spaces, namely B_s and B_{s+m}. As a result, the scalar ambiguity for individual maps does not influence the dimension of the homology of the mapping cone. However, when $m = 0$, different scalars would indeed make differences. See the end of Section 4 for an example of this subtlety.

Remark 1.4. The analogous result of the exact triangle (1.3) in Heegaard Floer theory was proved by Ozsváth-Szabó [OS08] using twisted coefficients, which is a crucial step towards proving the integral surgery formula in their setups. The proof cannot be applied to instanton theory directly. So in this paper, we go in a reversed way: we will use sutured instanton theory to prove Theorem 1.1 and derive Corollary 1.2 as a direct application. The strategy to prove Theorem 1.1 can be found in Section 3.1 and Section 3.2.

The analogs of $\pi^{\pm,s}$ in Heegaard Floer theory can be interpreted as cobordism maps associated to some particular spin^c structures. In instanton theory, there is a decomposition of cobordism maps along basic classes. However, currently such a decomposition is only known to exist for cobordisms whose first Betti number is zero. So for the moment let us assume the ambient 3-manifold Y is a rational homology sphere. For any integer m, there is a natural cobordism W_m from $-Y^3_{-m}(K)$ to

$-Y^3$. From [BS19, Section 1.2], there exists a decomposition of the cobordism map $I^\sharp(W_m)$ along basic classes

$$I^\sharp(W_m) = \sum_{s\in\mathbb{Z}} I^\sharp(W_m,[s]),$$

where $[s] \in H^2(W)$ denote the class that satisfies the equality

$$[s]([\bar{S}]) = 2s - m.$$

We make the following conjecture.

Conjecture 1.5. Suppose $K \subset Y$ is a null-homologous knot. Suppose $b_1(Y) = 0$ and $m \in \mathbb{Z}$ with $m \geqslant 2g(K) + 1$. Let A_s, B_s^-, $\pi^{\pm,s}$, W_m, $I^\sharp(W_m,[s])$ be defined as above. Then for any $s \in [\lfloor \frac{m-1}{2} \rfloor, \lfloor \frac{1-m}{2} \rfloor] \cap \mathbb{Z}$, there are commutative diagrams

$$\begin{tikzcd}
H(A_s) \arrow[r, hook] \arrow[d, "\pi^{-,s}"] & I^\sharp(-Y_{-m}(K)) \arrow[d, "I^\sharp(W_m,[s])"] \\
H(B_s) \arrow[r, "\cong"] & I^\sharp(-Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
H(A_s) \arrow[r, hook] \arrow[d, "\pi^{+,s}"] & I^\sharp(-Y_{-m}(K)) \arrow[d, "I^\sharp(W_m,[s+m])"] \\
H(B_s) \arrow[r, "\cong"] & I^\sharp(-Y)
\end{tikzcd}$$

The obstacle to obtain a decomposition of instanton cobordism map in general is one of the difficulties to export the original proof of the integral surgery formula in Heegaard Floer theory to instanton setup. To overcome this problem, we need to work with a suitable setup for which some kind of decompositions do exist. A good candidate is the sutured instanton theory. In sutured instanton theory, properly embedded surfaces induce $\mathbb{Z}$-gradings on the homology, and bypass maps relating different sutures are homogeneous with respect to such gradings. We have already used this setup to construct spin^c-like decompositions for the framed instanton homology of Dehn surgeries of knots, constructed bent complexes in instanton theory, and have established a large surgery formula in our previous work [LY22, LY21a, LY21b].

In this paper, to prove the integral surgery formula, we further study the relations between different sutures on the knot complement and establish some new exact triangles and commutative diagrams that may be of independent interests. Then these new and old algebraic structures relating different sutures enable us to apply the octahedral lemma to prove the desired integral surgery formula. It is worth mentioning that ultimately the whole proof in the current paper depends only on some most fundamental properties of Floer theory: the surgery exact triangle, the functoriality of the cobordism maps, and the adjunction inequality. This implies that the existence of the surgery formula is a born-in property of the Floer theory.

The surgery formula developed in the current paper is a powerful tool to study the Dehn surgeries along knots. It enables us to do explicit computations in many cases, even when the ambient 3-manifold has positive first Betti number. In a companion paper [LY], we will use the surgery formula and the techniques developed in this paper to derive many new applications and computations. We sketch the results as follows.

(1) We study the behavior of the integral surgery formula under the connected sum with a core knot in a lens space (whose complement is a solid torus) and then derive a rational surgery formula for framed instanton homology.
(2) We study the 0-surgery on a knot K inside S^3 or any other integral homology sphere instanton L-space. We derive a formula computing the nonzero grading part of $I^\sharp(S_0(K))$ with respect to the grading induced by the Seifert surface of the knot.

(3) We study the framed instanton homology of Dehn surgeries on instanton L-space knots and Floer simple knots, for which all complexes $A_s, B_s^\pm$ and the maps $\pi^\pm, \Xi_m$ are explicitly due to some 1-dimensional argument.
(4) We study almost L-space knots, *i.e.*, a non-L-space knot K so that there exists a nonzero integer n with $\dim I^\sharp(S^3_n(K)) = |n| + 2$. We prove a genus one almost L-space knot is either the figure-eight or the knot 5_2. We also show that almost L-space knots of genus at least 2 are fibered and strongly quasi-positive.
(5) We study some families of alternating knots. Using an inductive argument by oriented skein relation, we can describe their bent complexes explicitly and then the surgery formula applies routinely.
(6) Using the same technique as above, we also study the (non-zero) integral surgery of twisted Whitehead doubles. The results for whitehead doubles can also tell us the framed instanton Floer homology of the splicing of two knot complements in S^3, where one knot is either the trefoil or the figure-eight.
(7) We study (nonzero) integral surgeries on Boromean knots inside $\sharp^{2n} S^1 \times S^2$, which gives rise to circle bundles over surfaces with (nonzero) Euler numbers. In this case the the bent complexes A_s and $B_s^\pm$ can be computed directly and the maps $\pi^\pm$ between them can be fixed with the help of the H_1-action on the homology.

Organization. The paper is organized as follows. In Section 2, we introduce basic setups, the notations in sutured instanton homology, and deal with the scalar ambiguity mentioned in Remark 2.4. We also present some algebraic lemmas including the octahedral lemma in the derived category that are used in latter sections. In Section 3, we present the strategy to prove the integral surgery formula. We first restate the integral surgery formula using sutured instanton homology, and explain how to apply the octahedral lemma to prove it. Then we explain how to translate the integral surgery formula from the language of sutured instanton theory to the language of bent complex, which coincides with the discussions in the introduction. All the rest sections are devoted to prove the three exact triangles and three commutative diagrams that are involved in the octahedral lemma, *i.e.*, Equation (3.2) to Equation (3.7). In Section 4, we study the relation between the (-1)-Dehn surgery map associated to a curve intersecting the suture twice and the two natural bypass maps associated to that curve. This helps us to prove Equation (3.2) and Equation (3.5). In Section 5, Equation (3.3), Equation (3.6) and part of Equation (3.4) are proved. The rest two sections are devoted to prove Equation (3.4) and Equation 3.7), which is the most technical part of the paper. In Section 6 we prove some technical lemmas that are finally used in Section 7 to finish the proof.

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2. Basic setups

2.1. **Conventions.** If it is not mentioned, all manifolds are smooth, oriented, and connected. Homology groups and cohomology groups are with $\mathbb{Z}$ coefficients. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{F}_2$ for the field with two elements.

A knot $K \subset Y$ is called **null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Z})$, while it is called **rationally null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Q})$.

For any oriented 3-manifold M, we write $-M$ for the manifold obtained from M by reversing the orientation. For any surface S in M and any suture $\gamma \subset \partial M$, we write S and γ for the same surface and suture in $-M$, without reversing their orientations. For a knot K in a 3-manifold Y, we write $(-Y, K)$ for the induced knot in $-Y$ with induced orientation, called the **mirror knot** of K. The corresponding balanced sutured manifold is $(-Y\backslash N(K), -\gamma_K)$.

2.2. **Sutured instanton homology.** For any **balanced sutured manifold** (M,γ) [Juh06, Definition 2.2], Kronheimer-Mrowka [KM10, Section 7] constructed an isomorphism class of $\mathbb{C}$-vector spaces $SHI(M,\gamma)$. Later, Baldwin-Sivek [BS15, Section 9] dealt with the naturality issue and constructed (untwisted and twisted vesions of) projectively transitive systems related to $SHI(M,\gamma)$. We will use the twisted version, which we write as $\underline{\mathrm{SHI}}(M,\gamma)$ and call **sutured instanton homology**.

In this paper, when considering maps between sutured instanton homology, we can regard them as linear maps between actual vector spaces, at the cost that equations (or commutative diagrams) between maps only hold up to a nonzero scalar due to the projectivity. More detailed discussion on the projectivity can be found in the next subsection.

Moreover, there is a relative $\mathbb{Z}_2$-grading on $\underline{\mathrm{SHI}}(M,\gamma)$ obtained from the construction of sutured instanton homology, which we consider as a **homological grading**.

Definition 2.1. Suppose K is a knot in a closed 3-manifold Y. Let $Y(1) := Y\backslash B^3$ and let δ be a simple closed curve on $\partial Y(1) \cong S^2$. Let $Y\backslash N(K)$ be the knot complement and let Γ_μ be two oppositely oriented meridians of K on $\partial(Y\backslash N(K)) \cong T^2$. Define

$$I^\sharp(Y) := \underline{\mathrm{SHI}}(Y(1),\delta) \text{ and } \underline{\mathrm{KHI}}(Y,K) := \underline{\mathrm{SHI}}(Y\backslash N(K),\Gamma_\mu).$$

Remark 2.2. By the naturality results, we should specify the places of the removing ball, the neighborhood of the knot, and the sutures to define $I^\sharp(Y)$ and $\underline{\mathrm{KHI}}(Y,K)$. These data can be fixed by choosing a basepoint in Y or K. For simplicity, we omit those choices in the notations.

From now on, we will suppose $K \subset Y$ is a rationally null-homologous knot and fix some notations. Let μ be the meridian of K and pick a longitude λ (so that $\lambda\cdot\mu = 1$) to fix a framing of K. We will always assume $Y\backslash N(K)$ is irreducible, but many results still hold due to the following connected sum formula of sutured instanton homology [Li18a, Section 1.8]:

$$\underline{\mathrm{SHI}}(Y'\sharp Y\backslash N(K),\gamma) \cong I^\sharp(Y')\otimes \underline{\mathrm{SHI}}(Y\backslash N(K),\gamma).$$

Given coprime integers r and s, let $\Gamma_{r/s}$ be the suture on $\partial(Y\backslash N(K))$ consists of two oppositely oriented, simple closed curves of slope $-r/s$, with respect to the chosen framing (*i.e.* the homology of the curves are $\pm(-r\mu + s\lambda) \in H_1(\partial N(K))$). Pick S to be a minimal genus Seifert surface of K.

Convention. We will use p to denote the order of $[K] \in H_1(Y)$, *i.e.*, p is the minimal positive integer satisfying $p[K] = 0 \in H_1(Y)$. Let $q = \partial S\cdot\lambda$ and let $g = g(S)$ be the genus of S. When K is null-homologous, we always choose the Seifert framing $\lambda = \partial S$. In such case, we have $(p,q) = (1,0)$.

Remark 2.3. The meanings of p and q above are different from our previous papers [LY22, LY21b]. Before, we used $\hat{\mu}$ and $\hat{\lambda}$ to denote the meridian of the knot K and the preferred framing. When ∂S is connected, it is the same as the homological longitude λ in previous papers. Hence p and q in this paper should be q and q_0 in previous papers.

For simplicity, we use the bold symbol of the suture to represent the sutured instanton homology of the balanced sutured manifold with the reversed orientation:

$$\mathbf{\Gamma}_{r/s} := \underline{\mathrm{SHI}}(-(Y\backslash N(K)), -\Gamma_{r/s}).$$

When $(r,s) = (\pm 1, 0)$, we write $\Gamma_{r/s} = \Gamma_\mu$. When $s = \pm 1$, we write $\Gamma_n = \Gamma_{n/1} = \Gamma_{(-n)/(-1)}$. We also write $\mathbf{\Gamma}_\mu$ and $\mathbf{\Gamma}_n$ for the corresponding sutured instanton homologies.

Remark 2.4. Strictly speaking, the sutures corresponding to $(r,s) = (1,0)$ and $(-1,0)$ are not identical because the orientations are opposite. Since both sutures are on $\partial(Y\backslash N(K))$ of the same slope, they are isotopic. Moreover, we can choose a canonical isotopy by rotating the suture along the direction specified by the orientation of the knot. Due to discussion in Heegaard Floer theory [Sar15, Zem19] and the conjectural relation between Heegaard Floer theory and instanton theory [KM10], it is expected that rotating the suture back to the original position induces a nontrivial isomorphism of the sutured instanton homology. So we pick the canonical isotopy to be the minimal rotation of the suture. Hence we can abuse notations and write Γ_μ for both sutures. The same discussion also applies to the relation between $\Gamma_{n/1}$ and $\Gamma_{(-n)/(-1)}$.

By work of [Li19], the Seifert surface S induces a grading on $\mathbf{\Gamma}_{r/s}$. We always assume that S has minimal intersections with $\Gamma_{r/s}$. When the intersection number $\partial S \cdot (s\lambda - r\mu)$ is odd, then S induces a $\mathbb{Z}$-grading on $\mathbf{\Gamma}_{r/s}$. When $\partial S \cdot (s\lambda - r\mu)$ is even, we need to perform either a positive stabilization or negative stabilization on S to induce a $\mathbb{Z}$-grading, and the two gradings are related by an overall grading shift of 1. To get rid of stabilizations, we can equivalently regard that, in this case, the surface S induces a $(\mathbb{Z} + \frac{1}{2})$-grading. We write the graded part of $\mathbf{\Gamma}_{r/s}$ as

$$(\mathbf{\Gamma}_{r/s}, i) := \underline{\mathrm{SHI}}(-(Y\backslash N(K)), -\Gamma_{r/s}, S, i)$$

with $i \in \mathbb{Z}$ or $i \in \mathbb{Z} + \frac{1}{2}$, depending on the parity of the intersection number. From the construction of grading in [Li19], we have the following vanishing theorem due to the adjunction inequality.

Lemma 2.5. *We have* $(\mathbf{\Gamma}_{r/s}, i) = 0$ *when*

$$|i| > g + \frac{|rp - sq| - 1}{2}.$$

The bypass exact triangle for sutured instanton homology was introduced by Baldwin-Sivek in [BS22b, Section 4]. In [LY22, Section 4.2], we applied the triangle to sutures on knot complements and computed the grading shifts. We restate the results by notations introduced before.

Lemma 2.6. *For any* $n \in \mathbb{Z}$*, there are two graded bypass exact triangles*

$$\begin{tikzcd}
(\mathbf{\Gamma}_n, i+\frac{p}{2}) \arrow[rr, "\psi^n_{+,n+1}"] & & (\mathbf{\Gamma}_{n+1}, i) \arrow[dl, "\psi^{n+1}_{+,\mu}"] \\
& (\mathbf{\Gamma}_\mu, i-\frac{np-q}{2}) \arrow[ul, "\psi^\mu_{+,n}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{\Gamma}_n, i-\frac{p}{2}) \arrow[rr, "\psi^n_{-,n+1}"] & & (\mathbf{\Gamma}_{n+1}, i) \arrow[dl, "\psi^{n+1}_{-,\mu}"] \\
& (\mathbf{\Gamma}_\mu, i+\frac{np-q}{2}) \arrow[ul, "\psi^\mu_{-,n}"] &
\end{tikzcd}$$

where the maps are homogeneous with respect to the homological $\mathbb{Z}_2$*-gradings.*

Remark 2.7. The reason to use balanced sutured manifold with reversed orientation is because of the above bypass exact triangles.

Remark 2.8. Note that if we do not mention gradings, the above triangles and the results in the rest of this subsection also hold without the assumption that K is rationally-null homologous since the proofs only involve the neighborhood of $\partial(-Y\backslash N(K))$.

Corollary 2.9. *For any large enough integer n, we have the following properties for restrictions of maps.*

(1) *The map $\psi^n_{+,n+1}|(\mathbf{\Gamma}_n, i)$ is an isomorphism when $i \leqslant \frac{np-q-1}{2} - g$.*
(2) *The map $\psi^n_{-,n+1}|(\mathbf{\Gamma}_n, i)$ is an isomorphism when $i \geqslant g - \frac{np-q-1}{2}$.*
(3) *For any $g - \frac{np-q-1}{2} \leqslant i \leqslant \frac{np-q-1}{2} - g - p$, there is an isomorphism*
$$(\psi^n_{+,n+1})^{-1} \circ \psi^n_{-,n+1} : (\mathbf{\Gamma}_n, i) \xrightarrow{\cong} (\mathbf{\Gamma}_n, i+p).$$
(4) *The map $\psi^{-n}_{-,1-n}|(\mathbf{\Gamma}_{-n}, i)$ is an isomorphism when $i \leqslant \frac{(n-1)p+q-1}{2} - g$.*
(5) *The map $\psi^{-n}_{+,1-n}|(\mathbf{\Gamma}_{-n}, i)$ is an isomorphism when $i \geqslant g - \frac{(n-1)p+q-1}{2}$.*
(6) *For any $g - \frac{np+q-1}{2} \leqslant i \leqslant \frac{np+q+1}{2} - g - p$, there is an isomorphism*
$$(\psi^{-n}_{+,1-n})^{-1} \circ \psi^{-n}_{-,1-n} : (\mathbf{\Gamma}_{-n}, i) \xrightarrow{\cong} (\mathbf{\Gamma}_{-n}, i+p).$$

Proof. It is a combination of Lemma 2.5 and Lemma 2.6. □

Definition 2.10. The maps in Lemma 2.6 are called **bypass maps**. The ones with subscripts $+$ and $-$ are called **positive** and **negative bypass maps**, respectively. We will use $\pm$ to denote one of the bypass maps. For any integer n and any positive integer k, define
$$\Psi^n_{\pm,n+k} := \psi^{n+k-1}_{\pm,n+k} \circ \cdots \circ \psi^n_{\pm,n+1} : \mathbf{\Gamma}_n \to \mathbf{\Gamma}_{n+k}.$$

In [LY22, Section 4.4], we proved many commutative diagrams for bypasses maps, which we restate as follows by notations introduced before.

Lemma 2.11. *For any $n \in \mathbb{Z}$, we have the following commutative diagrams up to scalars.*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[r, "\psi^n_{+,n+1}"] \arrow[d, "\psi^n_{-,n+1}"'] & \mathbf{\Gamma}_{n+1} \arrow[d, "\psi^{n+1}_{-,n+2}"] \\
\mathbf{\Gamma}_{n+1} \arrow[r, "\psi^{n+1}_{+,n+2}"] & \mathbf{\Gamma}_{n+2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{n+2} \arrow[r, "\psi^{n+2}_{+,\mu}"] \arrow[d, "\psi^{n+2}_{-,\mu}"'] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^\mu_{+,n}"] \\
\mathbf{\Gamma}_\mu \arrow[r, "\psi^\mu_{-,n}"] & \mathbf{\Gamma}_n
\end{tikzcd}$$

Proof. The first diagram follows from [LY22, Lemma 4.33]. Note that the proof only used the functoriality of the contact gluing map and did not depend on the assumption that K is rationally null-homologous. The second diagram is obtained from the first diagram by changing the choice of the framed knot. Explicitly, let K' be the dual knot corresponding to $\mathbf{\Gamma}_{n+1}$. Let $\mu' = -(n+1)\mu + \lambda$ denote its meridian. Then $\lambda' = -\mu$ is a framing of K'. Applying the first diagram to K', we will obtain the second diagram for the original K. □

Lemma 2.12. *For any $n \in \mathbb{Z}$, we have the following commutative diagrams up to scalars*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{+,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{\Gamma}_\mu \arrow[ul, "\psi^\mu_{-,n}"] \arrow[ur, "\psi^\mu_{-,n+1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{-,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{\Gamma}_\mu \arrow[ul, "\psi^\mu_{+,n}"] \arrow[ur, "\psi^\mu_{+,n+1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{+,n+1}"] \arrow[dr, "\psi^n_{-,\mu}"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "\psi^{n+1}_{-,\mu}"] \\
& \mathbf{\Gamma}_\mu &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{-,n+1}"] \arrow[dr, "\psi^n_{+,\mu}"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "\psi^{n+1}_{+,\mu}"] \\
& \mathbf{\Gamma}_\mu &
\end{tikzcd}$$

There are more bypass triangles involving more complicated sutures, which are obtained from changing the choice of the framed knot as in the proof of Lemma 2.11.

Lemma 2.13. *For a knot $K \subset Y$ and $n \in \mathbb{Z}$, there are two graded bypass exact triangles*

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i + \frac{np-q}{2}) \arrow[rr, "\psi^{n-1}_{+,\frac{2n-1}{2}}"] & & (\mathbf{\Gamma}_{\frac{2n-1}{2}}, i) \arrow[dl, "\psi^{\frac{2n-1}{2}}_{+,n}"] \\
& (\mathbf{\Gamma}_n, i - \frac{(n-1)p-q}{2}) \arrow[ul, "\psi^n_{+,n-1}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i - \frac{np-q}{2}) \arrow[rr, "\psi^{n-1}_{-,\frac{2n-1}{2}}"] & & (\mathbf{\Gamma}_{\frac{2n-1}{2}}, i) \arrow[dl, "\psi^{\frac{2n-1}{2}}_{-,n}"] \\
& (\mathbf{\Gamma}_n, i + \frac{(n-1)p-q}{2}) \arrow[ul, "\psi^n_{-,n-1}"] &
\end{tikzcd}$$

Lemma 2.14. *For a knot $K \subset Y$ and $n \in \mathbb{Z}$, there are commutative diagrams up to scalars*

$$\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[r, "\psi^\mu_{+,n-1}"] \arrow[d, "\psi^\mu_{-,n-1}"'] & \mathbf{\Gamma}_{n-1} \arrow[d, "\psi^{n-1}_{+,\frac{2n-1}{2}}"] \\
\mathbf{\Gamma}_{n-1} \arrow[r, "\psi^{n-1}_{-,\frac{2n-1}{2}}"] & \mathbf{\Gamma}_{\frac{2n-1}{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n-1}{2}} \arrow[r, "\psi^{\frac{2n-1}{2}}_{+,n}"] \arrow[d, "\psi^{\frac{2n-1}{2}}_{-,n}"'] & \mathbf{\Gamma}_n \arrow[d, "\psi^n_{-,\mu}"] \\
\mathbf{\Gamma}_n \arrow[r, "\psi^n_{+,\mu}"] & \mathbf{\Gamma}_\mu
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[rr, "\psi^\mu_{+,n-1}"] & & \mathbf{\Gamma}_{n-1} \\
& \mathbf{\Gamma}_n \arrow[ul, "\psi^n_{-,\mu}"] \arrow[ur, "\psi^n_{+,n-1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[rr, "\psi^\mu_{-,n-1}"] & & \mathbf{\Gamma}_{n-1} \\
& \mathbf{\Gamma}_n \arrow[ul, "\psi^n_{+,\mu}"] \arrow[ur, "\psi^n_{-,n-1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_{n-1} \arrow[rr, "\psi^{\frac{2n-1}{2}}_{+,n-1}"] \arrow[dr, "\psi^{n-1}_{+,n}"'] & & \mathbf{\Gamma}_{\frac{2n-1}{2}} \arrow[dl, "\psi^{\frac{2n-1}{2}}_{-,n}"] \\
& \mathbf{\Gamma}_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{n-1} \arrow[rr, "\psi^{\frac{2n-1}{2}}_{-,n-1}"] \arrow[dr, "\psi^{n-1}_{-,n}"'] & & \mathbf{\Gamma}_{\frac{2n-1}{2}} \arrow[dl, "\psi^{\frac{2n-1}{2}}_{+,n}"] \\
& \mathbf{\Gamma}_n &
\end{tikzcd}$$

Remark 2.15. The choices of positive and negative bypass maps in Lemma 2.14 seem to be different from Lemma 2.11 and Lemma 2.12. But indeed they are the same up to changing the framed knot. In particular, the grading shifts match

Suppose α is a connected non-separating simple closed curve on $\partial(Y\backslash N(K))$, we can push α into the interior of $Y\backslash N(K)$ and apply the surgery exact triangle associated to the surgeries along α with respect to the framing induced by $\partial(Y\backslash N(K))$. According to [BS16a, Section 4], when α intersects the suture at two points, the 0-surgery along α corresponds to a 2-handle attachment along α and hence leads to the Dehn filling of $Y\backslash N(K)$ along α. We write

$$\mathbf{Y}_{\mathbf{r/s}} := I^{\sharp}(-Y_{-r/s}(K)),$$

and in particular

$$\mathbf{Y}_n := I^{\sharp}(-Y_{-n}(K)) \text{ and } \mathbf{Y} := I^{\sharp}(-Y).$$

Lemma 2.16. *For any $n \in \mathbb{Z}$, we have the following exact triangles.*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "H_n"] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "F_{n+1}"] \\
& \mathbf{Y} \arrow[ul, "G_n"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[rr, "A_{n-1}"] & & \mathbf{\Gamma}_{n-1} \arrow[dl, "B_{n-1}"] \\
& \mathbf{Y}_n \arrow[ul, "C_n"] &
\end{tikzcd}$$

Remark 2.17. From [BS16a, Section 4], we know the 0-surgery corresponds to a 2-handle attachement and a 1-handle attachement. Hence $\mathbf{Y}$ in above lemma is indeed $\underline{\mathrm{KHI}}^-(-Y,U)$, where U is the unknot in $-Y$ bounding an embedded disk. By [BS16a, Section 4], 1-handle attachement does not change the closure of the balanced sutured manifold, and then there is a canonical identification between $\underline{\mathrm{KHI}}(-Y,U)$ and $I^{\sharp}(-Y)$. Hence we can abuse the notations. The same discussion also applies to $\mathbf{Y}_{\mathbf{n}}$.

Furthermore, we proved the following properties in [LY22].

Lemma 2.18 ([LY22, Lemma 3.21 and Lemma 4.9])**.** *For any $n \in \mathbb{Z}$, we have the following commutative diagrams up to scalars*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^{n}_{+,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{Y} \arrow[ul, "G_n"] \arrow[ur, "G_{n+1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^{n}_{-,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{Y} \arrow[ul, "G_n"] \arrow[ur, "G_{n+1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^{n}_{+,n+1}"] \arrow[dr, "F_n"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "F_{n+1}"] \\
& \mathbf{Y} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^{n}_{-,n+1}"] \arrow[dr, "F_n"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "F_{n+1}"] \\
& \mathbf{Y} &
\end{tikzcd}$$

Lemma 2.19 ([LY22, Lemma 4.17, Proposition 4.26, Lemma 4.29 and Proposition 4.32])**.** *Let F_n and G_n be defined as in Lemma 2.16. Then for any large enough integer n, we have the following properties*

(1) The map G_{n-1} is zero and F_n is surjective. Moreover, for any grading

$$g-\frac{np-q-1}{2}\leqslant i_0\leqslant\frac{np-q-1}{2}-g-p+1,$$

the restriction of the map

$$F_n:\bigoplus_{i=0}^{p-1}(\mathbf{\Gamma}_n,i_0+i)\to\mathbf{Y}$$

is an isomorphism.

(2) The map F_{-n+1} is zero and G_{-n} is injective. Moreover, for any grading

$$g-\frac{np+q-1}{2}\leqslant i_0\leqslant\frac{np+q-1}{2}-g-p+1,$$

the map

$$\operatorname{Proj}\circ G_{-n}:\mathbf{Y}\to\bigoplus_{i=0}^{p-1}(\mathbf{\Gamma}_{-n},i_0+i),$$

is an isomorphism, where

$$\operatorname{Proj}:\mathbf{\Gamma}_{-n}\to\bigoplus_{i=0}^{p-1}(\mathbf{\Gamma}_{-n},i_0+i)$$

is the projection.

The following lemma is a special case of Proposition 4.1, which we will prove later.

Lemma 2.20. *For any $n\in\mathbb{Z}$, let the maps H_n and ψ^n_{n+1} be defined as in Lemma 2.16 and Lemma 2.6 respectively. Then there exist $c_1,c_2\in\mathbb{C}\backslash\{0\}$ so that*

$$H_n=c_1\psi^n_{+,n+1}+c_2\psi^n_{-,n+1}$$

2.3. **Fixing the scalars.** By construction, sutured instanton homology forms a projectively transitive system, which means all the spaces and maps between spaces are well-defined only up to nonzero scalars. When the balanced sutured manifold is obtained from a framed knot as in the last subsection, we can make some canonical choices to reduce the projective ambiguities.

Suppose $K\subset Y$ is a framed knot with the meridian μ and the framing λ. Fix a knot complement $Y\backslash N(K)$. Let S^1_1, S^1_2, S^3_3 be three circles. Take

$$\bar{Y}_\mu=(Y\backslash N(K))\cup_{\varphi_\mu}S^1_1\times(S^1_2\times S^1_3-D^2),$$

where φ_μ is an orientation reversing diffeomorphism sending S^1_1 to μ and $\partial(S^1_2\times S^1_3-D^2)$ to λ. By [KM10, Lemma 5.2], the manifold $\bar{Y}_\mu$ is a closure of $(Y\backslash N(K),\Gamma_\mu)$ with the distinguishing surface $T^2=S^1_2\times S^1_3$. The circle S^1_3 makes the pair $(\bar{Y}_\mu,S^1_3)$ admissible.

For the suture Γ_n, we can similarly take

$$\bar{Y}_n=(Y\backslash N(K))\cup_{\varphi_n}S^1_1\times(S^1_2\times S^1_3-D^2),$$

where φ_n is an orientation reversing diffeomorphism sending S^1_1 to $-n\mu+\lambda$ and $\partial(S^1_2\times S^1_3-D^2)$ to $-\mu$. Then $\bar{Y}_n$ is a closure of $(Y\backslash N(K),\Gamma_n)$.

For the suture $\Gamma_{\frac{2n-1}{2}}$, we can similarly take

$$\bar{Y}_{\frac{2n-1}{2}} = (Y\backslash N(K)) \cup_{\varphi_{\frac{2n-1}{2}}} S^1_1 \times (S^1_2 \times S^1_3 - D^2),$$

where $\varphi_{\frac{2n-1}{2}}$ is an orientation reversing diffeomorphism sending S^1_1 to $(1-2n)\mu + 2\lambda$ and $\partial(S^1_2 \times S^1_3 - D^2)$ to $-n\mu + \lambda$. Then $\bar{Y}_{\frac{2n-1}{2}}$ is a closure of $(Y\backslash N(K), \Gamma_{\frac{2n-1}{2}})$.

For $(Y(1), \delta)$, we can simply take the connected sum $Y\sharp S^1_1 \times S^1_2 \times S^1_3$, where the distinguishing surface is $S^1_2 \times S^1_3$. The circle S^1_3 makes the pair $(Y\sharp S^1_1 \times S^1_2 \times S^1_3, S^1_3)$ admissible. We reverse the orientations of the chosen closures when the orientations of the sutured manifolds are reversed. Note that we do not choose canonical closures for $Y_n(K)(1)$ since we only care about the dimension of its framed instanton homology.

After fixing the choices of closures, we can view $\mathbf{\Gamma}_n$ and $\mathbf{Y}$ as actual vector spaces and then elements in them are well-defined. Strictly speaking, we also need to choose extra data such as the metric and the perturbation on the closure to define the instanton Floer homology of the closure, but different choices of metrics and perturbations now induce a transitive system of vector spaces, from which we can construct an actual vector space. So we omit the discussion on those extra data.

The construction of bypass maps and surgery maps may not be realized as cobordism maps between the chosen closures, but the construction of the projectively transitive system guarantees the existence of such maps up to scalars. Now We make (non-canonical) choices of the maps to get rid of the scalar ambiguities in commutative diagrams mentioned in the last subsection.

By Lemma 2.19, we can pick a large enough integer n_0 so that G_{-n_0} is injective and F_{n_0} is surjective. Pick arbitrary representatives of maps

$$G_{-n_0}, F_{n_0}, \psi^{-n_0}_{+,\mu}, \psi^{-n_0}_{-,\mu}, \psi^{\mu}_{+,-n_0}, \psi^{\mu}_{-,-n_0}$$

and

$$\psi^n_{+,n+1}, \psi^{n-1}_{+,\frac{2n-1}{2}}$$

for all $n \in \mathbb{Z}$. By the first two commutative diagrams in Lemma 2.19, we can pick a representative for G_{-n_0+1} to satisfy the equation

$$G_{-n_0+1} = \psi^{-n_0}_{+,-n_0+1} \circ G_{-n_0}$$

and pick a representative for $\psi^{-n_0}_{-,-n_0+1}$ to satisfy the equation

$$G_{-n_0+1} = \psi^{-n_0}_{-,-n_0+1} \circ G_{-n_0}.$$

Then we can pick representatives of $\psi^n_{\pm,\mu}$ and $\psi^{\mu}_{\pm,n}$ for all n inductively to satisfy the commutative diagrams in Lemma 2.12 without introducing scalars. Similarly, the representative of $\psi^n_{-,n+1}$ for all n are determined by the first diagram in Lemma 2.11. The representatives of

$$\psi^{\frac{2n-1}{2}}_{-,n}, \psi^n_{+,n-1}, \psi^n_{-,n-1}, \psi^{n-1}_{-,\frac{2n-1}{2}}, \psi^{\frac{2n-1}{2}}_{+,n}$$

are determined by Lemma 2.14.

By the first commutative diagram in Lemma 2.19, we can pick representatives of G_n for all n inductively to satisfy the equality

$$G_n = \psi^{n-1}_{+,n} \circ G_{n-1}.$$

Now we verify the equality

$$G_n = \psi^{n-1}_{-,n} \circ G_{n-1}$$

We can verify this equality inductively and we only deal with the case that $n > n_0$. The other case is similar. *A priori*, Lemma 2.18 implies that for our choice of representatives, there exists $c \in \mathbb{C}\backslash\{0\}$ so that

$$G_n = c \cdot \psi_{-,n}^{n-1} \circ G_{n-1}.$$

We need to show that c has to equal to 1. Without loss of generality, we can assume that $G_{n-2} \neq 0$ and assume that $G_{n-2}(\alpha) \neq 0$ for some $\alpha \in \mathbf{Y}$. Then

$$\begin{aligned} G_n(\alpha) &= \psi_{+,n}^{n-1} \circ \psi_{-,n-1}^{n-2} \circ G_{n-2}(\alpha) \\ \text{(Lemma 2.11)} &= \psi_{-,n}^{n-1} \circ \psi_{+,n-1}^{n-2} \circ G_{n-2}(\alpha) \\ \text{(Inductive hypothesis)} &= \psi_{-,n}^{n-1} \circ G_{n-1}(\alpha) \end{aligned}$$

Hence $c = 1$.

Then we only need to choose representatives of F_n. Since we have already chosen F_{n_0}, we can pick F_{n_0+1} to satisfy the equation

$$F_{n_0+1} = \psi_{+,n_0+1}^{n_0} \circ F_{n_0}.$$

However, since we have already fixed the map $\psi_{-,n_0+1}^{n_0}$, Lemma 2.18 implies that there exists $c \in \mathbb{C}\backslash\{0\}$ so that

$$F_{n_0+1} \circ \psi_{-,n_0+1}^{n_0} = c \cdot F_{n_0}$$

for which we cannot simply take as 1. Using a similar argument as above, we can pick representatives of all F_n maps inductively so that

$$F_{n+1} = \psi_{+,n+1}^{n} \circ F_n \text{ and } F_{n+1} \circ \psi_{-,n+1}^{n} = c \cdot F_n.$$

Finally, we discuss scalars in Lemma 2.20. For any $n \in \mathbb{Z}$, we can pick a representative of H_n so that

$$H_n = \psi_{+,n+1}^{n} + c_n \cdot \psi_{-,n+1}^{n}$$

To get rid of this extra scalar c_n, we consider two cases.

Case 1. There exists an n so that $G_{n+1} \neq 0$ and $F_n \neq 0$. Take $\alpha \in \mathbf{Y}$ so that $G_{n+1}(\alpha) \neq 0$ and take $x = G_n(\alpha)$, we know that

$$\psi_{+,n+1}^{n}(x) + c_n \cdot \psi_{-,n+1}^{n}(x) = H_n(x) = 0.$$

From our choice of G_n, we also know that

$$\psi_{+,n+1}^{n}(x) = \psi_{-,n+1}^{n}(x) = G_{n+1}(\alpha) \neq 0.$$

Hence $c_n = -1$. Also from the assumption that $F_n \neq 0$, we can assume that $F_n(y) \neq 0$. As a result,

$$F_{n+1} \circ \psi_{+,n+1}^{n}(y) = F_n(y) \neq 0 \text{ and } F_{n+1} \circ \psi_{-,n+1}^{n}(y) = c \cdot F_n(y) \neq 0$$

On the other hand, we know that

$$\begin{aligned} (1 + c \cdot c_n) F_n(y) &= F_{n+1} \circ (\psi_{+,n+1}^{n} + c_n \cdot \psi_{-,n+1}^{n})(y) \\ &= F_{n+1} \circ H_n(y) \\ &= 0. \end{aligned}$$

As a result, we have $1 + c \cdot c_n = 0$. Hence $c = 1$.

Case 2. For any integer n, either $G_{n+1} = 0$ or $F_n = 0$. (For example, any knot inside S^3 falls into this case.) Take n_1 to be the largest integer so that $G_{n_1+1} \neq 0$. We know from the above argument that $c_n = -1$ for all $n \leqslant n_1$.

For $n > n_1$, define a grading perserving isomorphism

$$\iota_n : \mathbf{\Gamma}_n \to \mathbf{\Gamma}_n$$

so that for any $x \in (\mathbf{\Gamma}_n, i)$, we have

$$\iota_n(x) = c^j \cdot x \text{ where } j = \lfloor \frac{i - \frac{n-1}{2} - g}{p} \rfloor$$

Define a new H_n and F_n by

$$\widetilde{H}_n = \widetilde{H}_n = \psi^n_{+,n+1} - \psi^n_{-,n+1}, \text{ and } \widetilde{F}_n = F_n \circ \iota_n.$$

It is then straightforward to check that for any $n > n_1$, we have an exact triangle

$$\begin{tikzcd} \mathbf{\Gamma}_n \arrow[rr, "\widetilde{H}_n"] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "\widetilde{F}_{n+1}"] \\ & \mathbf{Y} \arrow[ul, "G_n"] & \end{tikzcd}$$

and equations

$$\widetilde{F}_{n+1} \circ \psi^n_{+,n+1} = \widetilde{F}_n \text{ and } \widetilde{F}_{n+1} \circ \psi^n_{-,n+1} = \widetilde{F}_n.$$

As a result, for $n > n_1$, we can use the new maps $\widetilde{H}_n$ and $\widetilde{F}_n$ to form the surgery triangle. From now on, we write them simply as H_n and F_n.

Convention. From the above discussion, when $K \subset Y$ is a rationally null-homologous knot, we can assume the first commutative diagram in Lemma 2.11 and all commutative diagrams in Lemma 2.12, Lemma 2.14 and Lemma 2.18 hold without introducing scalars. Moreover, we can assume $H_n = \psi^n_{+,n+1} - \psi^n_{-,n+1}$ for all n.

2.4. **Algebraic lemmas.** In this subsection, we introduce some lemmas in homological algebra. All graded vector spaces in this subsection are finite dimensional and over $\mathbb{C}$ and all maps are complex linear maps. For convenience, we will switch freely between long exact sequences and exact triangles.

From Section 2.2 , we know the sutured instanton homology is usually $\mathbb{Z} \oplus \mathbb{Z}_2$-graded, where we regard the $\mathbb{Z}_2$-grading as a homological grading. Many results in this subsection come from properties of the derived category of vector spaces over $\mathbb{C}$, for which people usually consider cochain complexes. However, for a $\mathbb{Z}_2$-graded space there is no difference between the chain complex and the cochain complex. Hence by saying a **complex** we mean a $\mathbb{Z}_2$-graded (co)chain complex, though all results apply to $\mathbb{Z}$-graded cochain complexes verbatim.

For a complex C and an integer n, we write C^n for its grading n part (under the natural map $\mathbb{Z} \to \mathbb{Z}_2$). With this notation, we suppose the differential d_C on C sends C^n to C^{n+1}. For any integer k, we write $C\{k\}$ for the complex obtained from C by the grading shift $C\{k\}^n = C^{n+k}$. We write $H(C, d_C)$ or $H(C)$ for the homology of a complex C with differential d_C. A $\mathbb{Z}_2$-graded vector space is regarded as a complex with the trivial differential.

For a chain map $f : C \to D$, we write $\text{Cone}(f)$ for the **mapping cone** of f, *i.e.*, the complex consisting of the space $D \oplus C\{1\}$ and the differential

$$d_{\text{Cone}(f)} := \begin{bmatrix} d_D & -f \\ 0 & -d_C \end{bmatrix}.$$

Then there is a long exact sequence

$$\cdots \to H(C) \xrightarrow{f} H(D) \xrightarrow{i} H(\mathrm{Cone}(f)) \xrightarrow{p} H(C)\{1\} \to \cdots$$

where i sends $x \in D$ to $(x, 0)$ and p sends $(x, y) \in D \oplus C\{1\}$ to $-y$. If differentials of C and D are trivial, then we know

$$H(\mathrm{Cone}(f)) \cong \ker(f) \oplus \mathrm{coker}(f). \tag{2.1}$$

Remark 2.21. Our definitions about mapping cones follow from [Wei94], which are different from those in [OS08, OS11].

Note that the derived category is a triangulated category, so it satisfies the octahedral lemma (for example, see [Wei94, Proposition 10.2.4]).

Lemma 2.22 (octahedral lemma)**.** *Suppose X, Y, Z, X', Y', Z' are $\mathbb{Z}_2$-graded vector spaces satisfying the following long exact sequences*

$$\begin{aligned} &\cdots \to X \xrightarrow{f} Y \xrightarrow{h} Z' \to X\{1\} \to \cdots \\ &\cdots \to X \xrightarrow{g\circ f} Z \xrightarrow{h'} Y' \xrightarrow{l'} X\{1\} \to \cdots \\ &\cdots \to Y \xrightarrow{g} Z \to X' \xrightarrow{l} Y\{1\} \to \cdots \end{aligned}$$

Then we have the fourth long exact sequence

$$\cdots \to Z' \xrightarrow{\psi} Y' \xrightarrow{\phi} X' \xrightarrow{h\{1\}\circ l} Z'\{1\} \to \cdots$$

such that the following diagram commutes

$$\begin{tikzcd}
& Y' \arrow[dr, dotted, "\phi"] \arrow[ddl] & \\
Z' \arrow[ur, dotted, "\psi"] \arrow[d] & & X' \arrow[ll, dotted, "h\{1\}\circ l"] \arrow[ddl] \\
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[u] \arrow[uul] \\
& Y \arrow[ur, "g"'] \arrow[uul] &
\end{tikzcd} \tag{2.2}$$

where we omit the grading shifts and the notations for maps h, l, j. We can also write (2.2) in another form so that there is enough room to write the maps

$$
\begin{tikzcd}
 & & & Z' \arrow[rr] \arrow[dr, dotted, "\psi"] & & X\{1\} \arrow[ddr, "f\{1\}"] & & \\
 & & Y \arrow[ur, "h"] \arrow[dr, "g"] & & Y' \arrow[ur, "l'"] \arrow[ddrr, dotted, "\phi"] & & & \\
 & & & Z \arrow[ur, "h'"] \arrow[drrr] & & & Y\{1\} \arrow[dr, "h\{1\}"] & \\
X \arrow[uurr, "f"] \arrow[urrr, "g\circ f"] & & & & & & X' \arrow[u, "l"] \arrow[r, dotted, "h\{1\}\circ l"] & Z'\{1\}
\end{tikzcd}
\tag{2.3}
$$

The map ψ and ϕ in (2.3) can be written explicitly as follows. By the long exact sequences in the assumption of Lemma 2.22, we know that Z', X', Y' are chain homotopic to the mapping cones $\mathrm{Cone}(f), \mathrm{Cone}(g), \mathrm{Cone}(g \circ f)$, respectively. Under such homotopies, we can write

$$
\begin{aligned}
\psi : Y \oplus X\{1\} &\to Z \oplus X\{1\} \\
\psi(y, x) &\mapsto (g(y), x)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi : Z \oplus X\{1\} &\to Z \oplus Y\{1\} \\
\phi(z, x) &\mapsto (z, f\{1\}(x))
\end{aligned}
$$

However, the chain homotopies are not canonical, and hence the maps ψ and ϕ are also not canonical. Thus, usually we cannot identify them with other given maps. Fortunately, with an extra $\mathbb{Z}$-grading, we may identify $H(\mathrm{Cone}(\phi))$ with $H(\mathrm{Cone}(\phi'))$ for another map $\phi' : Y' \to X'$.

First, We introduce the following lemma to deal with the projectivity of maps (*i.e.* maps well-defined only up to scalars). Note that the $\mathbb{Z}$-grading in the following lemma is not the homological grading used before.

Lemma 2.23. *Suppose X and Y are $\mathbb{Z}$-graded vector spaces and suppose $f, g : X \to Y$ are homogeneous maps with different grading shifts k_1 and k_2. Then* $\mathrm{Cone}(f + g)$ *is isomorphic to* $\mathrm{Cone}(c_1 f + c_2 g)$ *for any $c_1, c_2 \in \mathbb{C} \backslash \{0\}$.*

Proof. For simplicity, we can suppose $k_1 = 0$ and $k_2 = 1$. The proof for the general case is similar. For $i, j \in \mathbb{Z}$, we write (X, i) and (Y, j) for grading summands of X and Y, respectively. Suppose T is a automorphism of $X \oplus Y$ that acts by

$$\frac{c_2^i}{c_1^{i+1}} \mathrm{Id} \text{ on } (X, i) \text{ and } \frac{c_2^j}{c_1^j} \mathrm{Id} \text{ on } (Y, j).$$

Then T is an isomorphism between $\mathrm{Cone}(f + g)$ and $\mathrm{Cone}(c_1 f + c_2 g)$. $\square$

Then we state the lemma that relates the map ϕ in Lemma 2.22 to another map ϕ' constructed explicitly.

Lemma 2.24. *Suppose X, Y, Z, X', Y' are $\mathbb{Z}\oplus\mathbb{Z}_2$-graded vector spaces satisfying the following horizontal exact sequences.*

$$\begin{tikzcd}
Z \arrow[r, "h'"] \arrow[d, "="] & Y' \arrow[r, "l'"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\phi'=a'+b'"] & X\{1\} \arrow[d, "f\{1\}=a+b"] \\
Z \arrow[r, "\phi\circ h'=\phi'\circ h'"] & X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}$$

where the shift $\{1\}$ is for the $\mathbb{Z}_2$-grading. Suppose $\phi : Y' \to X'$ satisfies the two commutative diagrams and suppose $\phi' : Y' \to X'$ satisfies the left commutative diagram. Suppose l and l' are homogeneous with respect to the $\mathbb{Z}$-grading. Suppose $f\{1\} = a + b$ and $\phi' = a' + b'$ are sums of homogeneous maps with different grading shifts with respect to the $\mathbb{Z}$-grading. Moreover, suppose the following diagrams hold up to scalars.

$$\begin{tikzcd}
Y' \arrow[r, "l'"] \arrow[d, "a'"] & X\{1\} \arrow[d, "a"] \\
X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y' \arrow[r, "l'"] \arrow[d, "b'"] & X\{1\} \arrow[d, "b"] \\
X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}$$

Then there is an isomorphism between $H(\mathrm{Cone}(\phi))$ and $H(\mathrm{Cone}(\phi'))$.

Proof. Since ϕ and ϕ' share the same domain and codomain, it suffices to show that they have the same rank. Fixing inner products on Y' and X' so that we have orthogonal decompositions

$$Y' = \mathrm{Im}(h') \oplus Y'' \text{ and } X' = \mathrm{Im}(\phi\circ h') \oplus X''.$$

By commutativity, we know both ϕ and ϕ' send $\mathrm{Im}(h')$ onto $\mathrm{Im}(\phi\circ h')$. Hence if we choose basis with respect to the decompositions so that linear maps are represented by matrices (we use row vectors), then we have

$$\phi = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \text{ and } \phi' = \begin{bmatrix} A' & B' \\ 0 & C' \end{bmatrix},$$

where $A = A' : \mathrm{Im}(h') \to \mathrm{Im}(\phi\circ h')$ has full row rank. Then it suffices to show C and C' have the same row rank.

By the exactness at Y' and X', we know the restriction of l' on Y'' is an isomorphism between Y'' and $\mathrm{Im}(l')$ and the restriction of l on X'' is an isomorphism between X'' and $\mathrm{Im}(l)$. By commutativity, we know that both a and b send $\mathrm{Im}(l')$ to $\mathrm{Im}(l)$ and

$$\mathrm{rowrank}(C) = \mathrm{rank}(f\{1\}|\mathrm{Im}(l')) \text{ and } \mathrm{rowrank}(C') = \mathrm{rank}((c_1a + c_2b)|\mathrm{Im}(l'))$$

for some $c_1, c_2 \in \mathbb{C}\backslash\{0\}$. Since l and l' are homogeneous, there exist induced $\mathbb{Z}$-gradings on $\mathrm{Im}(l)$ and $\mathrm{Im}(l')$. The maps a and b are still homogeneous with different grading shifts with respect to these induced gradings. Then we can apply Lemma 2.23 to obtain that the ranks of the restrictions of $f\{1\} = a + b$ and $c_1a + c_2b$ on $\mathrm{Im}(l')$ are the same. □

3. Integral surgery formulae

Suppose $\widehat{K}$ is a (framed) rationally null-homologous knot in a closed 3-manifold $\widehat{Y}$. Given $m \in \mathbb{Z}$, suppose $\widehat{K}_{-m}$ is the dual knot in the manifold $\widehat{Y}_{-m}(\widehat{K})$ obtained from $\widehat{Y}$ by $(-m)$-surgery along $\widehat{K}$. In this section, we propose a conjectural formula calculating $I^\sharp(-\widehat{Y}_{-m}(\widehat{K}))$ analogous to the integral surgery formula in Heegaard Floer theory [OS08, OS11]. We also propose a strategy to prove this formula, but we could only realize the proof for $\widehat{Y}=S^3$ in Section 5 because we have to use the facts that $\dim_{\mathbb{C}} I^{\sharp}(S^3)=1$ and S^3 has an orientation-reversing involution in some steps.

3.1. **A formula for framed instanton homology.** In this subsection, we propose an integral surgery formula based on sutured instanton homology, and package it into the language of bent complexes in the next subsection.

Suppose $K\subset Y$ is a framed rationally null-homologous knot and we adapt notations introduced in Section 2.2. Define

$$\pi^{\pm}_{m,k}:=\Psi^{m+k}_{\pm,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{\mp,m+k}:\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}\to\mathbf{\Gamma}_{m+2k-1}$$

and write $\pi^{\pm,i}_{m,k}$ as the restriction of $\pi^{\pm}_{m,k}$ on $(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}},i)$. From Lemma 2.13 and Lemma 2.6 we can check that $\pi^{\pm}_{m,k}$ shifts grading by $\pm(mp-q)/2$. Then we can state the integral surgery formula.

Theorem 3.1. *Suppose m is a fixed integer such that $mp-q\neq 0$. Then for any large enough integer k, there exists an exact triangle*

$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[rr, "\pi^{+}_{m,k}+\pi^{-}_{m,k}"] & & \mathbf{\Gamma}_{m+2k-1} \arrow[dl] \\
& \mathbf{Y}_m \arrow[ul] &
\end{tikzcd}$$

Hence we have an isomorphism

$$\mathbf{Y}_m\cong H(\mathrm{Cone}(\pi^{+}_{m,k}+\pi^{-}_{m,k})).$$

Remark 3.2. Suppose μ and λ are the meridian and the longitude of the knot K. Then $mp-q\neq 0$ is equivalent to the fact that $-m\mu+\lambda$ is not isotopic to a connected component of the boundary of the Seifert surface. In particular, if K is null-homologous, this means $m\neq 0$.

In the rest of this subsection and in the next subsection, we state the strategy to prove Theorem 3.1, and leave the proofs of some propositions in the rest sections. An important step is to apply the octahedron axiom mentioned in Section 2.4 to the following diagram:

(3.1)
$$\begin{tikzcd}
& & \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[drr, dotted] \arrow[ddll] & & \\
\mathbf{Y}_m \arrow[urr, dotted] \arrow[d] & & & & \mathbf{\Gamma}_{m-1+2k} \arrow[llll, dotted] \arrow[dddll] \\
\mathbf{\Gamma}_{\mu} \arrow[rrrr] \arrow[drr] & & & & \mathbf{\Gamma}_{m-1+k}\oplus\mathbf{\Gamma}_{m-1+k} \arrow[uull] \arrow[u] \\
& & \mathbf{\Gamma}_{m-1} \arrow[uull] \arrow[urr] & &
\end{tikzcd}$$

To show the dotted exact triangle exist, We need to prove the following three exact triangles

$$\begin{tikzcd} I^\sharp(-S^3_{-m}(K)) \arrow[d] & \\ \mathbf{\Gamma}_\mu \arrow[dr] & \\ & \mathbf{\Gamma}_{m-1} \arrow[uul] \end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd} & \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[dl] & \\ \mathbf{\Gamma}_\mu \arrow[rr] & & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \arrow[ul] \end{tikzcd} \tag{3.3}$$

$$\begin{tikzcd} & \mathbf{\Gamma}_{m-1+2k} \arrow[ddl] \\ & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \arrow[u] \\ \mathbf{\Gamma}_{m-1} \arrow[ur] & \end{tikzcd} \tag{3.4}$$

and the following commutative diagram

$$\begin{tikzcd} \mathbf{\Gamma}_\mu \arrow[rr] \arrow[dr] & & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \\ & \mathbf{\Gamma}_{m-1} \arrow[ur] & \end{tikzcd} \tag{3.5}$$

Then the octahedral lemma will imply the existence of the dotted triangle and ensure that all diagrams in (3.1) other than exact triangles commute.

Then we will use Lemma 2.24 to identify the map coming from the octahedral lemma with $\pi^+_{m,k} + \pi^-_{m,k}$. We also need the following two extra commutative diagrams, where the maps other than $\pi^+_{m,k} + \pi^-_{m,k}$ come from (3.1).

$$\begin{tikzcd} \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[dr, "\pi^+_{m,k}+\pi^-_{m,k}"] & \\ & \mathbf{\Gamma}_{m-1+2k} \\ & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \arrow[u] \arrow[uul] \end{tikzcd} \tag{3.6}$$

$$
\begin{tikzcd}
 & \boldsymbol{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[ddl] \arrow[dr, "\pi^+_{m,k}+\pi^-_{m,k}"] & \\
 & & \boldsymbol{\Gamma}_{m-1+2k} \arrow[ddl] \\
\boldsymbol{\Gamma}_{\mu} \arrow[dr] & & \\
 & \boldsymbol{\Gamma}_{m-1} &
\end{tikzcd}
\tag{3.7}
$$

Indeed, when applying Lemma 2.24, we can prove some weaker commutative diagrams involving $\pi^{\pm}_{m,k}$ separately.

3.2. **A strategy of the proof.** In this subsection, we provide more details of the strategy mentioned in Section 3.1. For simplicity, we fix the scalar ambiguities of commutative diagrams as in Section 2.3. To write down the maps, we redraw the octahedral diagram (3.1) as follows:

$$
\begin{tikzcd}
 & & \mathbf{Y}_m \arrow[rr] \arrow[dr, dotted, "\psi"] & & \boldsymbol{\Gamma}_{\mu} \arrow[ddr, "\psi^{\mu}_{+,m-1}+\psi^{\mu}_{-,m-1}"] & \\
 & \boldsymbol{\Gamma}_{m-1} \arrow[ur] \arrow[dr, "(\Psi^{m-1}_{+,m-1+k}{,}\ \Psi^{m-1}_{-,m-1+k})"] & & \boldsymbol{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[ur, "l'"] \arrow[ddrr, dotted, "\phi"] & & \\
 & & \boldsymbol{\Gamma}_{m-1+k} \oplus \boldsymbol{\Gamma}_{m-1+k} \arrow[ur, "h'"'] \arrow[drrr, "\Psi^{m-1+k}_{-,m-1+2k}-\Psi^{m-1+k}_{+,m-1+2k}"'] & & & \boldsymbol{\Gamma}_{m-1} \\
\boldsymbol{\Gamma}_{\mu} \arrow[uur, "\psi^{\mu}_{+,m-1}+\psi^{\mu}_{-,m-1}"] \arrow[urr, "(\psi^{\mu}_{-,m-1+k}{,}\ \psi^{\mu}_{+,m-1+k})"'] & & & & & \boldsymbol{\Gamma}_{m-1+2k} \arrow[u, "l"']
\end{tikzcd}
\tag{3.8}
$$

where

$$h' = \psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} - \psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}.$$

The reader can compare (3.8) with (2.2) and (2.3). We omit the term corresponding to $Z'\{1\}$ because there is no enough room and the maps involving it are not important in our proof.

The first exact sequence of (3.8)

$$\boldsymbol{\Gamma}_{\mu} \xrightarrow{\psi^{\mu}_{+,m-1}+\psi^{\mu}_{-,m-1}} \boldsymbol{\Gamma}_{m-1} \to \mathbf{Y}_{\hat{\lambda}-m\hat{\mu}} \to \boldsymbol{\Gamma}_{\mu} \tag{3.9}$$

follows from the second exact triangle in Lemma 2.16. Though the map A_{m-1} may not be the same as the sum $\psi^{\mu}_{+,m-1}+\psi^{n}_{-,m-1}$, we can use the following proposition and Lemma 2.23 (another special case of Theorem 4.1) to identify $\mathrm{Cone}(A_{m-1})$ with $\mathrm{Cone}(\psi^{\mu}_{+,m-1}+\psi^{n}_{-,m-1})$. Here we use the assumption that $mp-q\neq 0$.

Proposition 3.3. *Suppose A_{n-1} is the map in Lemma 2.16. For any integer n, There exist scalars $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ such that*

$$A_{n-1} = c_1\psi^{\mu}_{+,m-1} + c_2\psi^{\mu}_{-,m-1}.$$

The exactness at

$$\mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k}$$

in the second and the third exact sequences are both special cases of the following proposition, which will be proved in Section 5.1 by diagram chasing.

Proposition 3.4. *Given $n \in \mathbb{Z}, k_0 \in \mathbb{N}_+$, suppose $c_0 \in \mathbb{C}\backslash\{0\}$ is the scalar such that the following equation holds*

$$\Psi^{n+k_0}_{+,n+2k_0} \circ \Psi^{n}_{-,n+k_0} = c_0\Psi^{n+k_0}_{-,n+2k_0} \circ \Psi^{n}_{+,n+k_0}.$$

Then for any c_1, c_2, c_3, c_4 satisfying the equation

$$c_1c_3 = -c_2c_4c_0,$$

the following sequence is exact

$$\mathbf{\Gamma}_n \xrightarrow{(c_1\Psi^{n}_{+,n+k_0}, c_2\Psi^{n}_{-,n+k_0})} \mathbf{\Gamma}_{n+k_0} \oplus \mathbf{\Gamma}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0} + c_4\Psi^{n+k_0}_{+,n+2k_0}} \mathbf{\Gamma}_{n+2k_0}$$

Remark 3.5. The exactness at the direct summand for the second exact sequence (the one involving $\mathbf{\Gamma}_{\hat{\mu}}$) might not be so clear from Proposition 3.4. Explicitly, we apply the proposition to the dual knot K' corresponding to Γ_{m+k} with framing $\lambda' = -\mu$ and $n = 0, k_0 = 1$. Note that by convention in Section 2.3, we have $c_0 = 1$.

The exactness at $\mathbf{\Gamma}_\mu$ and $\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}$ in the second exact sequence of (3.8)

$$\mathbf{\Gamma}_\mu \xrightarrow{(\psi^{\mu}_{-,m-1+k}, \psi^{\mu}_{+,m-1+k})} \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \xrightarrow{\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} - \psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}} \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \xrightarrow{l'} \mathbf{\Gamma}_\mu \tag{3.10}$$

will also be proved by diagram chasing. We can construct the map l' explicitly by the composition of bypass maps

$$l' := \psi^{m+k}_{-,\mu} \circ \psi^{\frac{2m+2k-1}{2}}_{+,m+k} = \psi^{m+k}_{+,\mu} \circ \psi^{\frac{2m+2k-1}{2}}_{-,m+k},$$

where the last equation follows from Lemma 2.14 and the convention in Section 2.3. The following proposition will be proved in Section 5.2 by diagram chasing.

Proposition 3.6. *Suppose l' is constructed as above. For any $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$, the following sequence is exact*

$$\mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \xrightarrow{c_3\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} + c_4\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}} \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \xrightarrow{l'} \mathbf{\Gamma}_\mu$$
$$\xrightarrow{(c_1\psi^{\mu}_{-,m-1+k}, c_2\psi^{\mu}_{+,m-1+k})} \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k}$$

Remark 3.7. In the proof of [LY21b, Theorem 3.23], we obtained a long exact sequence

$$\mathbf{\Gamma}_\mu \xrightarrow{(\psi^{\mu}_{+,n-1}, \psi^{\mu}_{-,n-1})} \mathbf{\Gamma}_{n-1} \oplus \mathbf{\Gamma}_{n-1} \to \mathbf{\Gamma}_{\frac{2n-1}{2}} \to \mathbf{\Gamma}_\mu$$

by the octahedral lemma. However, we did not know the two maps involving $\mathbf{\Gamma}_{\frac{2n-1}{2}}$ explicitly. Thus, the second exact sequence here is stronger than the one from octahedral lemma.

Remark 3.8. The reason that Proposition 3.6 holds for any choices of c_1, c_2, c_3, c_4 is because
$$\ker((c_1\psi^{\mu}_{-,m-1+k}, c_2\psi^{\mu}_{+,m-1+k})) = \ker(c_1\psi^{\mu}_{-,m-1+k}) \cap \ker(c_2\psi^{\mu}_{+,m-1+k})$$
and
$$\operatorname{Im}(c_3\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} + c_4\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}) = \operatorname{Im}(c_3\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}}) + \operatorname{Im}(c_4\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}),$$
where the right hand sides of the equations are independent of scalars.

The exactness at $\mathbf{\Gamma}_{m-1}$ and $\mathbf{\Gamma}_{m-1+2k}$ in the third exact sequence of (3.8)

$$\mathbf{\Gamma}_{m-1} \xrightarrow{(\Psi^{m-1}_{+,m-1+k},\Psi^{m-1}_{-,m-1+k})} \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \xrightarrow{\Psi^{m-1+k}_{-,m-1+2k} - \Psi^{m-1+k}_{+,m-1+2k}} \mathbf{\Gamma}_{m-1+2k} \xrightarrow{l} \mathbf{\Gamma}_{m-1} \tag{3.11}$$

is harder to prove since the map l cannot be constructed by bypass maps. We expect that there are many equivalent constructions of l and we will use the one for which the exactness is easiest to prove. Even so, we only prove the exactness with the assumption that k is large. See Section 7.2 for more details.

Proposition 3.9. *Suppose $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$ and suppose k_0 is a large integer. For any $n \in \mathbb{Z}$, there exists a map l such that the following sequence is exact*
$$\mathbf{\Gamma}_{n+k_0} \oplus \mathbf{\Gamma}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0} + c_4\Psi^{n+k_0}_{+,n+2k_0}} \mathbf{\Gamma}_{n+2k_0} \xrightarrow{l} \mathbf{\Gamma}_n \xrightarrow{(c_1\Psi^{n}_{+,n+k_0}, c_2\Psi^{n}_{-,n+k_0})} \mathbf{\Gamma}_{n+k_0} \oplus \mathbf{\Gamma}_{n+k_0}$$

Remark 3.10. In the first arXiv version of this paper, we only proved Proposition 3.9 for knots in S^3 because we had to use the fact that $\dim_{\mathbb{C}} I^{\sharp}(-S^3) = 1$ and S^3 has an orientation-reversing involution. The construction of l for knots in general 3-manifold are inspired by the original proof for S^3 and the proof in Section 7 is a generalization of the previous proof.

Remark 3.11. By the same reason in Remark 3.8, the coefficients in Proposition 3.9 are not important.

Then we consider the commutative diagrams mentioned in Section 3.1. By Lemma 2.6 and Lemma 2.12, we have
$$(\Psi^{m-1}_{+,m-1+k}, \Psi^{m-1}_{-,m-1+k}) \circ (\psi^{\mu}_{+,m-1} + \psi^{\mu}_{-,m-1}) = (\psi^{\mu}_{-,m-1+k}, \psi^{\mu}_{+,m-1+k}),$$
which verifies the commutative diagram in the assumption of the octahdral axiom.

Define
$$\phi' := \pi^{+}_{m,k} + \pi^{-}_{m,k} = \Psi^{m+k}_{+,m-1+2k} \circ \psi^{\frac{2m+2k-1}{2}}_{-,m+k} + \Psi^{m+k}_{-,m-1+2k} \circ \psi^{\frac{2m+2k-1}{2}}_{+,m+k}$$
By Lemma 2.13 and Lemma 2.14 with $n = m + k$, we have
$$\begin{aligned}
\phi' \circ h' =& (\Psi^{m+k}_{+,m-1+2k} \circ \psi^{\frac{2m+2k-1}{2}}_{-,m+k} + \Psi^{m+k}_{-,m-1+2k} \circ \psi^{\frac{2m+2k-1}{2}}_{+,m+k}) \circ (\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} - \psi^{m+k-1}_{+,\frac{2m+2k-1}{2}})\\
=& \Psi^{m+k}_{-,m-1+2k} \circ \psi^{\frac{2m+2k-1}{2}}_{+,m+k} \circ \psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} - \Psi^{m+k}_{+,m-1+2k} \circ \psi^{\frac{2m+2k-1}{2}}_{-,m+k} \circ \psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}\\
=& \Psi^{m+k}_{-,m-1+2k} \circ \psi^{m+k-1}_{-,m+k} - \Psi^{m+k}_{+,m-1+2k} \circ \psi^{m+k-1}_{+,m+k}\\
=& \Psi^{m-1+k}_{-,m-1+2k} - \Psi^{m-1+k}_{+,m-1+2k}
\end{aligned}$$
This verified the second commutative diagram mentioned in Section 3.1.

Finally, we state a weaker version of the third commutative diagram mentioned in Section 3.1, which is enough to apply Lemma 2.24. The following proposition will be proved in Section 7.4.

Proposition 3.12. *Suppose l' and l are the maps in Proposition 3.6 and Proposition 3.9. Then there are two commutative diagrams up to scalars.*

$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[r, "l'"] \arrow[d, "\pi^+_{m,k}"] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^\mu_{+,m-1}"] \\
\mathbf{\Gamma}_{m-1+2k} \arrow[r, "l"] & \mathbf{\Gamma}_{m-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[r, "l'"] \arrow[d, "\pi^-_{m,k}"] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^\mu_{-,m-1}"] \\
\mathbf{\Gamma}_{m-1+2k} \arrow[r, "l"] & \mathbf{\Gamma}_{m-1}
\end{tikzcd}$$

Proof of Theorem 3.1. We verified all assumptions of the octahedral lemma (Lemma 2.22) for the diagram (3.8). Hence there exists a map ϕ so that

$$\mathbf{Y}_m \cong H(\mathrm{Cone}(\phi)).$$

We also verified all assumptions of Lemma 2.24 for $\phi' = \pi^+_{m,k} + \pi^-_{m,k}$. Thus, we have

$$H(\mathrm{Cone}(\phi')) \cong H(\mathrm{Cone}(\phi)) \cong \mathbf{Y}_m.$$

Then the desired triangle in the theorem holds. □

3.3. **Reformulation by bent complexes.**

In this subsection, we restate Theorem 3.1 by the language of bent complexes introduced in [LY21b]. Suppose K is a rationally null-homologous knot in a closed 3-manifold Y. We still adapt notations and conventions in Section 2.2 and Section 2.3.

Putting bypass triangles in Lemma 2.6 for different n together, we obtain the following diagram:
(3.12)

$$\begin{tikzcd}[column sep=small]
\cdots & \mathbf{\Gamma}_{n+1} \arrow[l] \arrow[dr, "\psi^{n+1}_{+,\mu}"'] & & \mathbf{\Gamma}_n \arrow[ll, "\psi^n_{+,n+1}"'] \arrow[dr, "\psi^n_{+,\mu}"'] & & \mathbf{\Gamma}_{n-1} \arrow[ll, "\psi^{n-1}_{+,n}"'] \arrow[dr, "\psi^{n-1}_{+,\mu}"'] & & \mathbf{\Gamma}_{n-2} \arrow[ll, "\psi^{n-2}_{+,n-1}"'] & \cdots \arrow[l] \\
& \cdots & \mathbf{\Gamma}_\mu \arrow[ur, "\psi^\mu_{+,n}"] \arrow[dl, "\psi^\mu_{-,n-2}"'] & & \mathbf{\Gamma}_\mu \arrow[ur, "\psi^\mu_{+,n-1}"] \arrow[dl, "\psi^\mu_{-,n-1}"'] & & \mathbf{\Gamma}_\mu \arrow[ur, "\psi^\mu_{+,n-2}"] \arrow[dl, "\psi^\mu_{-,n}"'] & \cdots & \\
\cdots \arrow[r] & \mathbf{\Gamma}_{n-2} \arrow[rr, "\psi^{n-2}_{-,n-1}"'] & & \mathbf{\Gamma}_{n-1} \arrow[rr, "\psi^{n-1}_{-,n}"'] \arrow[ul, "\psi^{n-1}_{-,\mu}"'] & & \mathbf{\Gamma}_n \arrow[rr, "\psi^n_{-,n+1}"'] \arrow[ul, "\psi^n_{-,\mu}"'] & & \mathbf{\Gamma}_{n+1} \arrow[r] \arrow[ul, "\psi^{n+1}_{-,\mu}"'] & \cdots
\end{tikzcd}$$

where the $\mathbb{Z}$-grading shift of $\psi^\mu_{\pm,k} \circ \psi^k_{\pm,\mu}$ is $\pm p$ for any $k \in \mathbb{Z}$. From (3.12), we constructed in [LY21b, Section 3.4] two spectral sequences $\{E_{r,+}, d_{r,+}\}_{r \geqslant 1}$ and $\{E_{r,-}, d_{r,-}\}_{r \geqslant 1}$ from $\mathbf{\Gamma}_\mu$ to $\mathbf{Y}$, where $d_{r,\pm}$ is roughly

$$\psi^k_{\pm,\mu} \circ (\Psi^k_{\pm,k+r})^{-1} \circ \psi^\mu_{\pm,k+r} \text{ for any } k \in \mathbb{Z}. \tag{3.13}$$

The composition with the inverse map is well-defined on the r-th page and the independence of k follows from Lemma 2.12. The $\mathbb{Z}$-grading shift of $d_{r,\pm}$ is $\pm rp$. By fixing an inner product on $\mathbf{\Gamma}_\mu$, we then lifted those spectral sequences to two differentials d_+ and d_- on $\mathbf{\Gamma}_\mu$ so that

$$H(\mathbf{\Gamma}_\mu, d_+) \cong H(\mathbf{\Gamma}_\mu, d_-) \cong \mathbf{Y}.$$

In such way, the inverses of $\Psi^k_{\pm,k+r}$ are also well-defined, which we will use freely later.

Then we propose an integral surgery formula for $\mathbf{Y}_m$ using differentials d_+ and d_- on $\mathbf{\Gamma}_\mu$. To state the formula, we introduce the following notations.

Definition 3.13 ([LY21b, Construction 3.27 and Definition 5.12])**.** For any integer s, define complexes

$$B^{\pm}(s) := (\bigoplus_{k\in\mathbb{Z}}(\mathbf{\Gamma}_\mu, s+kp), d_{\pm}), \quad B^{+}(\geqslant s) := (\bigoplus_{k\geqslant 0}(\mathbf{\Gamma}_\mu, s+kp), d_{+}),$$
$$\text{and } B^{-}(\leqslant s) := (\bigoplus_{k\leqslant 0}(\mathbf{\Gamma}_\mu, s+kp), d_{-}).$$

Define

$$I^{+}(s) : B^{+}(\geqslant s) \to B^{+}(s) \text{ and } I^{-}(s) : B^{-}(\leqslant s) \to B^{-}(s)$$

to be the inclusion maps. We also write the same notation for the induced map on homology.

Remark 3.14. By Lemma 2.5, we know nontrivial gradings of $\mathbf{\Gamma}_\mu$ are finite. Then for any large enough integer s_0 such that

$$s - s_0 p < -g - \frac{p-1}{2} \text{ and } s + s_0 p > g + \frac{p-1}{2},$$

we have

$$B^{+}(s) = B^{+}(\leqslant s - s_0 p) \text{ and } B^{-}(s) = B^{-}(\geqslant s + s_0 p).$$

In such case, $I^{+}(s - s_0 p)$ and $I^{-}(s + s_0 p)$ are identities.

By splitting the diagram (3.12) into $\mathbb{Z}$-gradings, we can calculate homologies of complexes defined in Definition 3.13.

Proposition 3.15. *Suppose $n \in \mathbb{N}_+$ and i is a grading. Fix an inner product on $\mathbf{\Gamma}_n$. If $i > g + (p-1)/2 - np$, then there exists a canonical isomorphism*

$$H(B^{+}(\geqslant i)) \cong (\mathbf{\Gamma}_n, i + \frac{(n-1)p - q}{2}).$$

If $i < -g - (p-1)/2 + np$, then there exists a canonical isomorphism

$$H(B^{-}(\leqslant i)) \cong (\mathbf{\Gamma}_n, i - \frac{(n-1)p - q}{2}).$$

Proof. The proof is similar to that of [LY21b, Lemma 5.13]. Following the notations in [LY21b, (3.9) and (3.10)], if

$$i > \hat{i}^{\mu}_{max} - nq = g + (p-1)/2 - np,$$

then $\mathbf{\Gamma}_0^{i,+} = 0$ (the corresponding grading summand of $\mathbf{\Gamma}_0$) and the isomorphism follows from the convergence theorem of the unrolled spectral squence [LY21b, Theorem 2.4] (see also [Boa99, Theorem 6.1]). Note that the unrolled spectral sequence induces a filtration on $\mathbf{\Gamma}_n$ and the homology is canonically isomorphic to the direct sum of all associated graded objects of the filtration. Then we use the inner product to identify the direct sum with the total space $\mathbf{\Gamma}_n$. The other statement holds by the same reason. □

Definition 3.16 ([LY21b, Construction 3.27 and Definition 5.12])**.** For any integer s, define the **bent complex**

$$A(s) := (\bigoplus_{k\in\mathbb{Z}}(\mathbf{\Gamma}_\mu, s+kp), d_s),$$

where for any element $x \in (\mathbf{\Gamma}_\mu, s+kp)$,

$$d_s(x) = \begin{cases} d_+(x) & k > 0, \\ d_+(x) + d_-(x) & k = 0, \\ d_-(x) & k < 0. \end{cases}$$

Define
$$\pi^+(s): A(s) \to B^+(s) \text{ and } \pi^-(s): A(s) \to B^-(s)$$
by
$$\pi^+(s)(x) = \begin{cases} x & k \geqslant 0, \\ 0 & k < 0, \end{cases} \text{ and } \pi^-(s)(x) = \begin{cases} x & k \leqslant 0, \\ 0 & k > 0, \end{cases}$$
where $x \in (\mathbf{\Gamma}_\mu, s + kp)$. Define
$$\pi^\pm : \bigoplus_{s \in \mathbb{Z}} A(s) \to \bigoplus_{s \in \mathbb{Z}} B^\pm(s)$$
by putting $\pi^\pm(s)$ together for all s . We also use the same notation for the induced map on homology.

Remark 3.17. Similar to Remark 3.14, by Lemma 2.5, we know nontrivial gradings of $\mathbf{\Gamma}_\mu$ are finite. Then for any large enough integer s_0 such that $s_0 > g + (p-1)/2$, we have
$$A(s_0) = B^-(s_0) \text{ and } A(-s_0) = B^+(-s_0).$$
In such case, $\pi^-(s_0)$ and $\pi^+(-s_0)$ are identities.

Now we state the integral surgery formula in the above setup.

Theorem 3.18. *Suppose m is a fixed integer such that $mp - q \neq 0$. Then there exists a grading preserving isomorphism*
$$\Xi_m : \bigoplus_{s \in \mathbb{Z}} H(B^+(s)) \xrightarrow{\cong} \bigoplus_{s \in \mathbb{Z}} H(B^-(s + mp - q))$$
so that
$$\mathbf{Y}_m \cong H\Big(\operatorname{Cone}(\pi^- + \Xi_m \circ \pi^+ : \bigoplus_{s \in \mathbb{Z}} H(A(s)) \to \bigoplus_{s \in \mathbb{Z}} H(B^-(s)))\Big).$$

Proof. By Remark 3.17, we only need to consider the maps $\pi^\pm(s)$ for $|s|$ smaller than a fixed integer. For such s, we can apply the following proposition.

Proposition 3.19 ([LY21b, Proposition 3.28])**.** *Fix $m, s \in \mathbb{Z}$ such that $|s| \leqslant g + \frac{p-1}{2}$. For any large integer k, fix inner products on $\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}$ and $\mathbf{\Gamma}_{m-1+2k}$. Then there exist $s_1, s_2^+, s_2^-, s_3^+, s_3^- \in \mathbb{Z}$ so that the following diagram commutes*

$$\begin{tikzcd} H(A(s)) \arrow[r, "\pi^\pm(s)"] \arrow[d, "\cong"] & H(B^\pm(s)) \arrow[d, "\cong"] \\ (\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, s_1) \arrow[r, "\pi^{\pm, s_1}_{m,k}"] & (\mathbf{\Gamma}_{m-1+2k}, s_3^\pm) \end{tikzcd}$$

where $\pi^{\pm,s_1}_{m,k}$ are maps defined in Section 3.1 that factor through $(\mathbf{\Gamma}_{m+k}, s_2^\pm)$.

Remark 3.20. The maps $\pi^\pm(s)$ factor through $I^\pm(s)$ constructed in Definition 3.13. We write
$$\pi^\pm(s) = I^+(s) \circ \pi^{\pm,\prime}(s).$$

This corresponds to the factorization about $(\mathbf{\Gamma}_{m+k}, s_2^{\pm})$ in Proposition 3.19 (we fix an inner product on $\mathbf{\Gamma}_{m+k}$ to apply Proposition 3.15), *i.e.*, the following diagrams commute

$$\begin{tikzcd}
H(A(s)) \arrow[r, "\pi^{+,\prime}(s)"] \arrow[d, "\cong"] & H(B^+(\geqslant s)) \arrow[r, "I^+(s)"] \arrow[d, "\cong"] & H(B^+(s)) \arrow[d, "\cong"] \\
(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, s_1) \arrow[r, "\psi_{-,m+k}^{\frac{2m+2k-1}{2}}"] & (\mathbf{\Gamma}_{m+k}, s_2^+) \arrow[r, "\Psi_{+,m-1+2k}^{m+k}"] & (\mathbf{\Gamma}_{m-1+2k}, s_3^+)
\end{tikzcd}$$

$$\begin{tikzcd}
H(A(s)) \arrow[r, "\pi^{-,\prime}(s)"] \arrow[d, "\cong"] & H(B^-(\leqslant s)) \arrow[r, "I^-(s)"] \arrow[d, "\cong"] & H(B^-(s)) \arrow[d, "\cong"] \\
(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, s_1) \arrow[r, "\psi_{+,m+k}^{\frac{2m+2k-1}{2}}"] & (\mathbf{\Gamma}_{m+k}, s_2^-) \arrow[r, "\Psi_{-,m-1+2k}^{m+k}"] & (\mathbf{\Gamma}_{m-1+2k}, s_3^-)
\end{tikzcd}$$

From the calculation in [LY21b, Remark 3.29] (we replace n and l there by $m+k$ and $k-1$, and note that there is a typo about sign in the first arXiv version of [LY21b]), the difference of the grading shifts is

$$s_3^+ - s_3^- = (m+k-(k-1)-1)p - q = mp - q.$$

Note that the notations in this paper and [LY21b] are different (*c.f.* Remark 2.3).

Then we can construct the isomorphism

$$\Xi_m : \bigoplus_{s\in\mathbb{Z}} H(B^+(s)) \xrightarrow{\cong} \bigoplus_{s\in\mathbb{Z}} H(B^-(s+mp-q))$$

by identifying both $H(B^+(s))$ and $H(B^-(s+mp-q))$ with $(\mathbf{\Gamma}_{m-1+2k}, s_3^+)$. Since we only care about integer s with $|s|$ smaller than a fixed integer, we can take a large enough integer k to construct the isomorphism Ξ_m. *A priori*, this isomorphism depends on inner products on

$$\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, \mathbf{\Gamma}_{m-1+2k} \text{ and } \mathbf{\Gamma}_{m+k}.$$

Then we have

$$H(\mathrm{Cone}(\pi^- + \Xi_m \circ \pi^+)) \cong H(\mathrm{Cone}(\pi_{m,k}^- + \pi_{m,k}^+)) \cong \mathbf{Y}_m,$$

where the last isomorphism comes from Theorem 3.1. □

Remark 3.21. Theorem 3.18 is slightly weaker than Theorem 3.1. Indeed, when we use the integral surgery formula to calculate surgeries on the Boromean knot in the companion paper [], we have to study the $H_1(Y)$ action on sutured instanton homology, where $Y = \sharp^{2n} S^1 \times S^2$ is the ambient manifold of the knot. This action vanishes on $\mathbf{\Gamma}_\mu$ so vanishes on the bent complex. But it is nonvanishing on $\mathbf{\Gamma}_{m+k}$ and $\mathbf{\Gamma}_{m-1+2k}$ and we use this information to realize the computation. This issue for the bent complex might be resolved by introducing some E_0-pages for differentials d_+ and d_- so that the action is nontrivial on E_0-pages.

3.4. **A formula for instanton knot homology.** The third exact sequence (3.11) implies

$$\mathbf{\Gamma}_{m-1} \cong H(\mathrm{Cone}(\Psi_{-,m-1+2k}^{m-1+k} - \Psi_{-,m-1+2k}^{m-1+k}))$$

for any large enough integer k. We can remove the minus sign by Lemma 2.23. In this subsection, we restate this result by the language of bent complexes. The formula is inspired by Eftekhary's formula for knot Floer homology $\widehat{HFK}$ [Eft18, Proposition 1.5] (see also Hedden-Levine's work [HL21]). Since m can be any integer, we replace $m-1$ by m.

Theorem 3.22. *Suppose $m, j \in \mathbb{Z}$. Let*

$$j^{+} = j - \frac{(m-1)p-q}{2} \text{ and } j^{-} = j + \frac{(m-1)p-q}{2}.$$

Then there exists an isomorphism

$$\Xi_{m,j}' : H(B^{+}(j^{+})) \xrightarrow{\cong} H(B^{-}(j^{-}))$$

so that

$$(\mathbf{\Gamma}_m, j) \cong H\bigg(\mathrm{Cone}(I^{-}(j^{-}) + \Xi_{m,j}' \circ I^{+}(j^{+}) : H(B^{-}(\leqslant j^{-})) \oplus H(B^{+}(\geqslant j^{+})) \to H(B^{-}(j^{-})))\bigg).$$

Proof. As mentioned before, we have

$$\mathbf{\Gamma}_m \cong H(\mathrm{Cone}(\Psi_{-,m+2k}^{m+k} - \Psi_{+,m+2k}^{m+k})) \cong H(\mathrm{Cone}(\Psi_{-,m+2k}^{m+k} + \Psi_{+,m+2k}^{m+k}))$$

for any large enough integer k.

Since bypass maps are homogeneous, the above mapping cone splits into $\mathbb{Z}$-gradings (or $(\mathbb{Z}+\frac{1}{2})$-gradings). Hence we can use it to calculate $(\mathbf{\Gamma}_m, j)$. By Lemma 2.6, the corresponding spaces are

$$(\mathbf{\Gamma}_{m+k}, j + \frac{kp}{2}) \oplus (\mathbf{\Gamma}_{m+k}, j - \frac{kp}{2}) \text{ and } (\mathbf{\Gamma}_{m+2k}, j).$$

By Proposition 3.15, by fixing inner product on $\mathbf{\Gamma}_{m+k}$, we know that

$$(\mathbf{\Gamma}_{m+k}, j + \frac{kp}{2}) \cong H(B^{-}(\leqslant j^{-})) \text{ for } j + kp < -g - \frac{p-1}{2} + (m+k)p$$

and

$$(\mathbf{\Gamma}_{m+k}, j - \frac{kp}{2}) \cong H(B^{-}(\geqslant j^{+})) \text{ for } j - \frac{kp}{2} > g + \frac{p-1}{2} - (m+k)p.$$

Since m is fixed, when k is large enough, we know that any j with $(\mathbf{\Gamma}_m, j)$ nontrivial satisfies the above inequalities. By Proposition 3.15 again (fixing an inner product on $\mathbf{\Gamma}_{m+2k}$) and Remark 3.14, for k large enough, we know that

$$(\mathbf{\Gamma}_{m+2k}, j) \cong H(B^{-}(j^{-})) \cong H(B^{+}(j^{+})).$$

By unpackaging the construction of differentials d_+ and d_- in [LY21b, Section 3.4], we know that and the restrictions of maps $\Psi_{-,m+2k}^{m}$ and $\Psi_{+,m+2k}^{m}$ on the corresponding gradings coincide the maps induced by the inclusions $I^{-}(j^{-})$ and $I^{+}(j^{+})$ under the canonical isomorphisms, respectively.

Suppose

$$\Xi_{m,j}' : H(B^{+}(j^{+})) \xrightarrow{\cong} H(B^{-}(j^{-}))$$

is the isomorphism obtained from identifying both spaces to the corresponding grading summand of $\mathbf{\Gamma}_{m+2k}$. Note that it depends on inner products on $\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_{m+k}$ and $\mathbf{\Gamma}_{m+2k}$. Then we know that

$$\begin{aligned}(\mathbf{\Gamma}_m, j) \cong & H(\mathrm{Cone}(\Psi^{m+k}_{-,m+2k} + \Psi^{m+k}_{+,m+2k}|(\mathbf{\Gamma}_{m+k}, j+\frac{kp}{2}) \oplus (\mathbf{\Gamma}_{m+k}, j-\frac{kp}{2}))) \\ \cong & H(\mathrm{Cone}(I^-(j^-) + \Xi'_{m,j} \circ I^+(j^+))).\end{aligned}$$

□

4. Dehn surgery and bypass maps

In this section, we prove a generalization of Lemma 2.20 and Lemma 3.3.

Suppose (M,γ) is a balanced sutured manifold and $\alpha \subset \partial M$ is a connected simple closed curve that intersects the suture γ twice. There are two natural bypass arcs associated to α, which lead to two bypass triangles (*c.f.* [BS22b, Section 4])

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\gamma) \arrow[rr, "\psi_\pm"] & & \underline{\mathrm{SHI}}(-M,-\gamma_2) \arrow[dl] \\ & \underline{\mathrm{SHI}}(-M,-\gamma_3) \arrow[ul] & \end{tikzcd}$$

where γ_2 and γ_3 are the sutures coming from bypass attachments. Note the two bypass exact triangles involve the same set of balanced sutured manifolds but with different maps between them. Let (M_0,γ_0) be obtained from (M,γ) by attaching a contact 2-handle along α. From [BS16b, Section 3.3], it is shown that a closure of $(-M_0,-\gamma_0)$ coincides with a closure of the sutured manifold obtained from $(-M,-\gamma)$ by 0-surgery along α with respect to the surface framing. Hence there is also a surgery exact triangle (*c.f.* [LY22, Lemma 3.21])

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\gamma) \arrow[rr, "H_\alpha"] & & \underline{\mathrm{SHI}}(-M,-\gamma_2) \arrow[dl] \\ & \underline{\mathrm{SHI}}(-M_0,-\gamma_0) \arrow[ul] & \end{tikzcd}$$

The map H_α is related to the bypass maps $\psi_\pm$ as follows:

Proposition 4.1. *There exist $c_1, c_2 \in \mathbb{C}\backslash\{0\}$, so that*

$$H_\alpha = c_1\psi_+ + c_2\psi_-.$$

Remark 4.2. The proof of Proposition 4.1 is obtained during the discussion with John A. Baldwin and Steven Sivek.

Proof of Proposition 4.1. Let $A \subset \partial M$ be a tubular neighborhood of $\alpha \subset \partial M$. Pushing the interior of A into the interior of M to make it a properly embedded surface. By a standard argument in [Hon00], we can assume that a collar of ∂M is equipped with a product contact structure so that γ is (isotopic to) the dividing set, α is a Legendrian curve, A is in the contact collar, and A is a convex surface with Legendrian boundary that separates a standard contact neighborhood of α off M. The convex decomposition of M along A yields two pieces

$$M = M' \underset{A}{\cup} V,$$

where M' is diffeomorphic to M and V is the contact neighborhood of α. It is straightforward to check that after rounding the conners the contact structure near the boundary of M' is still a product contact structure with $\partial M'$ a convex boundary. Let γ' be the dividing set on $\partial M'$. Also, after rounding the conners the contact structure on $V \cong S^1 \times D^2$, we suppose ∂V is a convex surface with dividing set being the union of two connected simple closed curve on ∂V of slope -1. When viewing V as the complement of an unknot in S^3, the dividing set coincides with the suture $\Gamma_1 \subset V$, so from now on we call it Γ_1. By the construction of gluing map in [Li18b], there exists a map

$$G_1 : \underline{\mathrm{SHI}}(-M', -\gamma') \otimes \underline{\mathrm{SHI}}(-V, -\Gamma_1) \to \underline{\mathrm{SHI}}(-M, -\gamma).$$

As in [Li18b], the map G_1 comes from attaching contact handles to $(M', \gamma') \sqcup (V, \Gamma_1)$ to recover the gluing along A. From [Li19, Proposition 1.4], we know that

$$\underline{\mathrm{SHI}}(-V, -\Gamma_1) \cong \mathbb{C}.$$

Note that M' and M are both equipped with the product contact structure near the boundary. From the functoriality of the contact gluing map in [Li18b], we know that G_1 is an isomorphism. Now both the (-1)-surgery along a push off of α and the bypass attachments can be thought of as happening in the piece V. Note that the result of both (-1)-surgery and the bypass attachments for Γ_1 is Γ_2. Hence we have the following commutative diagram.

$$\begin{tikzcd}
\underline{\mathrm{SHI}}(-M', -\gamma') \otimes \underline{\mathrm{SHI}}(-V, -\Gamma_1) \arrow[r, "G_1"] \arrow[d, "\mathrm{Id} \otimes \widehat{H}_\alpha"'] & \underline{\mathrm{SHI}}(-M, -\gamma) \arrow[d, "H_\alpha"] \\
\underline{\mathrm{SHI}}(-M', -\gamma') \otimes \underline{\mathrm{SHI}}(-V, -\Gamma_2) \arrow[r, "G_2"] & \underline{\mathrm{SHI}}(-M, -\gamma_2)
\end{tikzcd} \tag{4.1}$$

where $\widehat{H}_\alpha$ denotes the surgery map for the manifold V and G_2 is the gluing map obtained by attaching the same set of contact handles as G_1. A similar commutative diagram holds when replacing H_α and $\widehat{H}_\alpha$ by $\psi_\pm$ and

$$\hat{\psi}_\pm : \underline{\mathrm{SHI}}(-V, -\Gamma_1) \to \underline{\mathrm{SHI}}(-V, -\Gamma_2)$$

in (4.1), respectively.

Since G_1 is an isomorphism, to obtain a relation between H_α and $\psi_\pm$, it suffices to understand the relation between $\widehat{H}_\alpha$ and $\hat{\psi}_\pm$. From [Li19, Proposition 1.4], we know that

$$\underline{\mathrm{SHI}}(-V, -\Gamma_2) \cong \mathbb{C}^2.$$

Moreover, the meridian disk of V induces a $(\mathbb{Z} + \frac{1}{2})$ grading on $\underline{\mathrm{SHI}}(-V, -\Gamma_2)$ and we have

$$\underline{\mathrm{SHI}}(-V, -\Gamma_2) \cong \underline{\mathrm{SHI}}(-V, -\Gamma_2, \frac{1}{2}) \oplus \underline{\mathrm{SHI}}(-V, -\Gamma_2, -\frac{1}{2}),$$

with

$$\underline{\mathrm{SHI}}(-V, -\Gamma_2, \frac{1}{2}) \cong \underline{\mathrm{SHI}}(-V, -\Gamma_2, -\frac{1}{2}) \cong \mathbb{C}.$$

Let

$$\mathbf{1} \in \underline{\mathrm{SHI}}(-V, -\Gamma_1) \cong \mathbb{C}$$

be a generator. In [Li19, Section 4.3] it is shown that

$$\hat{\psi}_-(\mathbf{1}) \in \underline{\mathrm{SHI}}(-V, -\Gamma_2, \frac{1}{2}) \text{ and } \hat{\psi}_+(\mathbf{1}) \in \underline{\mathrm{SHI}}(-V, -\Gamma_2, -\frac{1}{2})$$

are nonzero. Also, when viewing V as the complement of the unknot U, there is an exact triangle

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-V,-\Gamma_1) \arrow[rr, "\widehat{H}_\alpha"] & & \underline{\mathrm{SHI}}(-V,-\Gamma_2) \arrow[dl, "F_2"] \\ & I^\sharp(-S^3) \arrow[ul, "G_1"] & \end{tikzcd} \tag{4.2}$$

as in Lemma 2.16. Comparing the dimensions of the spaces in (4.2), we have $G_1 = 0$ and $\widehat{H}_\alpha$ is injective. From the fact that $\tau_I(U) = 0$, we know from [GLW19, Corollary 3.5] that

$$F_2|\underline{\mathrm{SHI}}(-V,-\Gamma_2,\frac{1}{2}) \neq 0 \text{ and } F_2|\underline{\mathrm{SHI}}(-V,-\Gamma_2,-\frac{1}{2}) \neq 0,$$

By the exactness in (4.2), we have $\ker(F_2) = \mathrm{Im}(\widehat{H}_\alpha)$ and then $\widehat{H}_\alpha(\mathbf{1})$ is not in $\underline{\mathrm{SHI}}(-V,-\Gamma_2,\pm\frac{1}{2})$, *i.e.*, it is a linear combination of generators of $\underline{\mathrm{SHI}}(-V,-\Gamma_2,\pm\frac{1}{2})$. Hence we know that there are $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ so that

$$\widehat{H}_\alpha(\mathbf{1}) = c_1\hat{\psi}_+(\mathbf{1}) + c_2\hat{\psi}_-(\mathbf{1}).$$

Then the proposition follows from the commutative diagram (4.1). □

In Remark 1.3, we discussed the ambiguity coming from scalars. It is worth to mention that such ambiguity already exists in instanton theory. For example, if M is the complement of a knot $K \subset S^3$ and γ consists of two meridians of the knot, which we denote by Γ_μ, we can pick α to be a curve on $\partial(S^3\backslash N(K))$ of slope $-n$. Then we have a surgery triangle:

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\Gamma_\mu) \arrow[rr, "H_n"] & & \underline{\mathrm{SHI}}(-M,-\Gamma_{n-1}) \arrow[dl] \\ & I^\sharp(-S^3_{-n}(K)) \arrow[ul] & \end{tikzcd}$$

Note this triangle is not the one from Floer's original exact triangle, but the one with slight modification on the choice of 1-cycles inside the 3-manifold that represents the second Stiefel-Whitney class of the relevant $SO(3)$-bundle; see [BS21, Section 2.2] for more details. Floer's original exact triangle, on the other hand, yields a different triangle

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-S^3\backslash N(K),-\Gamma_\mu) \arrow[rr, "H'_n"] & & \underline{\mathrm{SHI}}(-S^3\backslash N(K),-\Gamma_{n-1}) \arrow[dl] \\ & I^\sharp(-S^3_{-n}(K),\mu) \arrow[ul] & \end{tikzcd}$$

where $\mu \subset -S^3_{-n}(K)$ denotes a meridian of the knot. Note the difference between H_α and H'_α is that they come from the same cobordism but the $SO(3)$-bundles over the cobordism are different. The local argument to prove Proposition 4.1 works for both H_α and H'_α. Hence there exists non-zero complex numbers c_1, c_2, c'_1, c'_2 so that

$$H_\alpha = c_1\psi^\mu_{+,n} + c_2\psi^\mu_{-,n-1} \text{ and } H'_\alpha = c'_1\psi^\mu_{+,n} + c'_2\psi^\mu_{-,n-1}$$

where the maps

$$\psi^\mu_{\pm,n-1} : \underline{\mathrm{SHI}}(-S^3\backslash N(K),-\Gamma_\mu) \to \underline{\mathrm{SHI}}(-S^3\backslash N(K),-\Gamma_{n-1})$$

are the two related bypass maps. When $n \neq 0$, these two bypass maps have different grading shifting behavior, so by Lemma 2.23, different choice of non-zero coefficients does not change the dimensions of kernel and cokernel of the map. Hence we conclude that for $n \neq 0$,

$$I^{\sharp}(-S^3_{-n}(K),\mu) \cong I^{\sharp}(-S^3_{-n}(K)).$$

However, when $n = 0$, the two bypass maps $\psi^{\mu}_{\pm,n-1}$ are both grading preserving, so the coefficients matters, *i.e.*, $I^{\sharp}(-S^3_0(K),\mu)$ and $I^{\sharp}(-S^3_0(K))$ might have different dimensions. Indeed, it is observed by Baldwin-Sivek [BS21] that for what they called as W-shaped knots (which is clearly a non-empty class, *e.g.* the figure-8 knot [BS22a, Proposition 10.4]), these two framed instanton homologies have dimensions differed by 2.

5. Some exactness by diagram chasing

5.1. **At the direct summand.** In this subsection, we prove Proposition 3.4 by diagram chasing. For convenience, we restate it as follows.

Proposition 5.1. *Given $n \in \mathbb{Z}, k_0 \in \mathbb{N}_+$, suppose $c_0 \in \mathbb{C}\backslash\{0\}$ is the scalar such that the following equation holds*

$$\Psi^{n+k_0}_{+,n+2k_0} \circ \Psi^{n}_{-,n+k_0} = c_0 \Psi^{n+k_0}_{-,n+2k_0} \circ \Psi^{n}_{+,n+k_0}. \tag{5.1}$$

Then for any c_1,c_2,c_3,c_4 satisfying the equation

$$c_1c_3 = -c_2c_4c_0,$$

the following sequence is exact

$$\mathbf{\Gamma}_n \xrightarrow{(c_1\Psi^n_{+,n+k_0},c_2\Psi^n_{-,n+k_0})} \mathbf{\Gamma}_{n+k_0} \oplus \mathbf{\Gamma}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0}+c_4\Psi^{n+k_0}_{+,n+2k_0}} \mathbf{\Gamma}_{n+2k_0}$$

Proof. For simplicity, we only prove the proposition for $n = 0$. The proof for any general n is similar (replacing all $\mathbf{\Gamma}_m$ below by $\mathbf{\Gamma}_{n+m}$ and modifying the notations for bypass maps). Also, we adapt conventions in Section 2.3 and suppose

$$c_0 = c_1 = c_2 = c_3 = 1, c_4 = -1.$$

The proof for general scalars can be obtained similarly.

We prove the proposition by induction on k_0. We will use the exactness in Lemma 2.6 and the commutative diagrams in Lemma 2.12 and Lemma 2.11 for many times. For simplicity, we will use them without mentioning the lemmas.

First we assume $k_0 = 1$. The proposition reduces to

$$\ker(\psi^1_{-,2} - \psi^1_{+,2}) = \mathrm{Im}((\psi^0_{+,1},\psi^0_{-,1})).$$

The assumption (5.1) implies

$$\ker(\psi^1_{-,2} - \psi^1_{+,2}) \supset \mathrm{Im}((\psi^0_{+,1},\psi^0_{-,1})).$$

Then we prove

$$\ker(\psi^1_{-,2} - \psi^1_{+,2}) \subset \mathrm{Im}((\psi^0_{+,1},\psi^0_{-,1})).$$

Suppose

$$(x_1,x_2) \in \ker(\psi^1_{-,2} - \psi^1_{+,2}), i.e., \psi^1_{-,2}(x_1) - \psi^1_{+,2}(x_2) = 0.$$

Then we have

$$\psi^1_{+,\mu}(x_1) = \psi^2_{+,\mu} \circ \psi^1_{-,2}(x_1) = \psi^2_{+,\mu} \circ \psi^1_{+,2}(x_2) = 0.$$

By exactness, there exists $y \in \mathbf{\Gamma}_0$ so that $\psi^0_{+,1}(y) = x_1$. Then

$$\psi^1_{+,2} \circ \psi^0_{-,1}(y) = \psi^1_{-,2} \circ \psi^0_{+,1}(y) = \psi^1_{-,2}(x_1) \text{ and } \psi^1_{+,2}(x_2 - \psi^0_{-,1}(y)) = 0.$$

By exactness, there exists $z \in \mathbf{\Gamma}_\mu$ so that

$$\psi^\mu_{+,1}(z) = x_2 - \psi^0_{-,1}(y).$$

Let $y' = y + \psi^\mu_{+,0}(z)$. Then

$$\psi^0_{+,1}(y') = \psi^0_{+,1}(y) = x_1$$

and

$$\psi^0_{-,1}(y') = \psi^0_{-,1}(y) + \psi^0_{-,1} \circ \psi^\mu_{+,0}(z) = \psi^0_{-,1}(y) + \psi^\mu_{+,1}(z) = x_2,$$

which concludes the proof for $k_0 = 1$.

Suppose the proposition holds for $k_0 = k$. We prove it also holds for $k_0 = k + 1$. The proof is similar to the case for $k_0 = 1$. Again by assumption (5.1), we have

$$\ker(\Psi^{k+1}_{-,2k+2} - \Psi^{k+1}_{+,2k+2}) \supset \operatorname{Im}((\Psi^0_{+,k+1}, \Psi^0_{-,k+1})).$$

Then we prove

$$\ker(\Psi^{k+1}_{-,2k+2} - \Psi^{k+1}_{+,2k+2}) \subset \operatorname{Im}((\Psi^0_{+,k+1}, \Psi^0_{-,k+1})).$$

Suppose

$$(x_1, x_2) \in \ker(\Psi^{k+1}_{-,2k+2} - \Psi^{k+1}_{+,2k+2}), \, i.e., \, \Psi^{k+1}_{-,2k+2}(x_1) - \Psi^{k+1}_{+,2k+2}(x_2) = 0.$$

Then we have

$$\psi^{k+1}_{+,\mu}(x_1) = \psi^{2k+2}_{+,\mu} \circ \Psi^{k+1}_{-,2k+2}(x_1) = \psi^{2k+2}_{+,\mu} \circ \Psi^{k+1}_{+,2k+2}(x_2) = 0.$$

By exactness, there exists $y_1 \in \mathbf{\Gamma}_k$ so that $\psi^k_{+,k+1}(y_1) = x_1$. By a similar reason, there exists $y_2 \in \mathbf{\Gamma}_k$ so that $\psi^k_{-,k+1}(y_2) = x_2$. The goal is to prove

$$\Psi^k_{-,2k}(y_1') = \Psi^k_{+,2k}(y_2')$$

for some modifications y_1' and y_2' of y_1 and y_2 as for y' in the case of $k_0 = 1$. Then the induction hypothesis will imply that there exists $w \in \mathbf{\Gamma}_0$ so that

$$\Psi^0_{+,k}(w) = y_1' \text{ and } \Psi^0_{-,k}(w) = y_2'.$$

Hence we will have

$$\Psi^0_{+,k+1}(w) = \psi^k_{+,k+1}(y_1') = x_1 \text{ and } \Psi^0_{-,k+1}(w) = \psi^k_{-,k+1}(y_2') = x_2.$$

This will conclude the proof for $k_0 = k + 1$.

Now we start to construct y_1'. We have

$$\begin{aligned}
\psi^{2k+1}_{+,2k+2}(\Psi^{k+1}_{+,2k+1}(x_2) - \Psi^k_{-,2k+1}(y_1)) &= \psi^{2k+1}_{+,2k+2}(\Psi^{k+1}_{+,2k+1}(x_2) - \Psi^k_{-,2k+1}(y_1)) \\
&= \Psi^{k+1}_{+,2k+2}(x_2) - \psi^{2k+1}_{+,2k+2} \circ \Psi^k_{-,2k+1}(y_1) \\
&= \Psi^{k+1}_{-,2k+2}(x_1) - \Psi^{k+1}_{-,2k+2}(x_1) \\
&= 0.
\end{aligned}$$

By exactness, there exists $z_1 \in \mathbf{\Gamma}_\mu$ so that

$$\psi^\mu_{+,2k+1}(z_1) = \Psi^{k+1}_{+,2k+1}(x_2) - \Psi^k_{-,2k+1}(y_1).$$

Let $y_1' = y_1 + \psi^\mu_{+,k}(z_1)$. Then

$$\psi^k_{+,k+1}(y_1') = \psi^k_{+,k+1}(y_1) = x_1$$

and

$$\begin{aligned}\Psi^k_{-,2k+1}(y_1') &= \Psi^k_{-,2k+1}(y_1) + \Psi^k_{-,2k+1} \circ \psi^\mu_{+,k}(z_1)\\ &= \Psi^k_{-,2k+1}(y_1) + \psi^\mu_{+,2k+1}(z_1)\\ &= \Psi^{k+1}_{+,2k+1}(x_2),\end{aligned}$$

Then we start to construct y_2'. We have

$$\begin{aligned}\psi^{2k}_{-,2k+1}(\Psi^k_{-,2k}(y_1') - \Psi^k_{+,2k}(y_2)) &= \Psi^k_{-,2k+1}(y_1') - \psi^{2k}_{-,2k+1} \circ \Psi^k_{+,2k}(y_2)\\ &= \Psi^k_{-,2k+1}(y_1') - \Psi^{k+1}_{-,2k+1}(x_2)\\ &= 0.\end{aligned}$$

By exactness, there exists $z_2 \in \mathbf{\Gamma}_\mu$ so that

$$\psi^\mu_{-,2k}(z_2) = \Psi^k_{-,2k}(y_1') - \Psi^k_{+,2k}(y_2).$$

Let $y_2' = y_2 + \psi^\mu_{-,k}(z_2)$. Then

$$\psi^k_{-,k+1}(y_2') = \psi^k_{-,k+1}(y_2) = x_2$$

and

$$\begin{aligned}\Psi^k_{+,2k}(y_2') &= \Psi^k_{+,2k}(y_2) + \Psi^k_{+,2k} \circ \psi^\mu_{-,k}(z_2)\\ &= \Psi^k_{+,2k}(y_2) + \psi^\mu_{-,2k}(z_2)\\ &= \Psi^k_{-,2k}(y_1'),\end{aligned}$$

Then we have the following commutative diagrams

$$\begin{tikzcd}
 & & x_1 \in \mathbf{\Gamma}_{k+1} \arrow[ddrrr, "-"] & & & \\
 & y_1' \in \mathbf{\Gamma}_k \arrow[ur, "+"] \arrow[drr, "-"] & & & & \\
\mathbf{\Gamma}_0 \arrow[ur, "+"] \arrow[ddr, "-"'] & & & \mathbf{\Gamma}_{2k} \arrow[dr, "-"] & & \mathbf{\Gamma}_{2k+2} \\
 & & & & \mathbf{\Gamma}_{2k+1} \arrow[ur, "+"] & \\
 & y_2' \in \mathbf{\Gamma}_k \arrow[uurr, "+"] \arrow[dr, "-"] & & & & \\
 & & x_2 \in \mathbf{\Gamma}_{k+1} \arrow[uurr, "+"] & & &
\end{tikzcd}$$

By the induction hypothesis, there exists $w \in \mathbf{\Gamma}_0$ so that

$$\Psi^0_{+,k}(w) = y_1' \text{ and } \Psi^0_{-,k}(w) = y_2',$$

which concludes the proof for $k_0 = k + 1$. □

Remark 5.2. By similar arguments, we can prove the following sequence is exact for any $k_1, k_2 \in \mathbb{N}_+$

$$\mathbf{\Gamma}_n \xrightarrow{(c_1\Psi^n_{+,n+k_1}, c_2\Psi^n_{-,n+k_2})} \mathbf{\Gamma}_{n+k_1} \oplus \mathbf{\Gamma}_{n+k_2} \xrightarrow{c_3\Psi^{n+k_1}_{-,n+k_1+k_2}+c_4\Psi^{n+k_2}_{+,n+k_1+k_2}} \mathbf{\Gamma}_{n+k_1+k_2},$$

where the scalars are determined by $c_1c_3 = -c_2c_4c_0$ and c_0 comes from the following equation

$$\Psi^{n+k_2}_{+,n+k_1+k_2} \circ \Psi^n_{-,n+k_2} = c_0\Psi^{n+k_1}_{-,n+k_0+k_2} \circ \Psi^n_{+,n+k_1}.$$

5.2. **The second exact triangle.** In this subsection, we prove Proposition 3.6 by diagram chasing. For convenience, we restate it as follows, which is a little stronger than the previous version. Applying the proposition to the dual knot of $\mathbf{\Gamma}_{m+k}$ with framing $-\mu$ and setting $n = -1$ will recover Proposition 3.6.

Proposition 5.3. *Suppose*

$$l' = \psi^\mu_{+,n-1} \circ \psi^{n+1}_{+,\mu} = c_0\psi^\mu_{-,n-1} \circ \psi^{n+1}_{-,\mu}$$

for some $c_0 \in \mathbb{C}\backslash\{0\}$. Then for any $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$, the following sequence is exact

$$\mathbf{\Gamma}_n \oplus \mathbf{\Gamma}_n \xrightarrow{c_3\psi^n_{-,n+1}+c_4\psi^n_{+,n+1}} \mathbf{\Gamma}_{n+1} \xrightarrow{l'} \mathbf{\Gamma}_{n-1} \xrightarrow{(c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})} \mathbf{\Gamma}_n \oplus \mathbf{\Gamma}_n.$$

Proof. We adapt the conventions in Section 2.3. We will use Lemma 2.6, Lemma 2.11 and Lemma 2.12 without mentioning them. We prove the exactness at $\mathbf{\Gamma}_{n-1}$ first. We have

$$\psi^{n-1}_{\pm,n} \circ l' = \psi^{n-1}_{\pm,n} \circ \psi^\mu_{\pm,n-1} \circ \psi^{n+1}_{\pm,\mu} = 0.$$

Hence

$$\ker((c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})) \supset \operatorname{Im}(l').$$

Then we prove

$$\ker((c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})) \subset \operatorname{Im}(l').$$

Suppose

$$x \in \ker((c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})) = \ker(\psi^{n-1}_{-,n}) \cap \ker(\psi^{n-1}_{+,n}).$$

By exactness, there exists $y \in \mathbf{\Gamma}_\mu$ so that $\psi^\mu_{+,n-1}(y) = x$. Then we have

$$\psi^\mu_{+,n}(y) = \psi^{n-1}_{-,n} \circ \psi^\mu_{+,n-1}(y) = \psi^{n-1}_{-,n}(x) = 0.$$

By exactness, there exists $z \in \mathbf{\Gamma}_{n+1}$ so that $\psi^{n+1}_{+,\mu}(z) = y$. Thus, we have $l'(z) = x$, which concludes the proof for the exactness at $\mathbf{\Gamma}_{n-1}$.

Then we prove the exactness at $\mathbf{\Gamma}_{n+1}$. Similarly by exactness, we have

$$\ker(l') \supset \operatorname{Im}(c_3\psi^n_{-,n+1} + c_4\psi^n_{+,n+1}) = \operatorname{Im}(\psi^n_{-,n+1}) + \operatorname{Im}(\psi^n_{+,n+1}).$$

Suppose $x \in \ker(l')$. If $\psi^{n+1}_{+,\mu}(x) = 0$, then by the exactness, we know $x \in \operatorname{Im}(\psi^n_{+,n+1})$. If $\psi^{n+1}_{+,\mu}(x) \neq 0$, then by the exactness, there exists $y \in \mathbf{\Gamma}_n$ so that

$$\psi^n_{+,\mu}(y) = \psi^{n+1}_{+,\mu}(x).$$

Then we know

$$x - \psi^n_{-,n+1}(y) \in \ker(\psi^{n+1}_{+,\mu}) = \operatorname{Im}(\psi^n_{+,n+1}).$$

Thus, we have

$$x \in \operatorname{Im}(\psi^n_{-,n+1}) + \operatorname{Im}(\psi^n_{+,n+1}),$$

which concludes the proof for the exactness at $\mathbf{\Gamma}_{n+1}$. $\square$

6. Some technical constructions

6.1. **Filtrations.** In this subsection, we study some filtrations on $\mathbf{Y}$ and $\mathbf{\Gamma}_\mu$ that will be important in latter sections. We still adapt conventions in Section 2.3.

Lemma 6.1. *The maps G_n in Lemma 2.16 lead to a filtration on $\mathbf{Y}$: for large enough integer n_0,*

$$0=\ker G_{-n_0}\subset\cdots\subset\ker G_n\subset\ker G_{n+1}\subset\cdots\subset\ker G_{n_0}=\mathbf{Y}.$$

Proof. It follows from Lemma 2.19 that when n_0 is large enough we have

$$0=\ker G_{-n_0}\text{ and }\ker G_{n_0}=\mathbf{Y}.$$

It follows from Lemma 2.18 that for any $n\in\mathbb{Z}$,

$$\ker G_n\subset\ker G_{n+1}.$$

□

Lemma 6.2. *For any $n\in\mathbb{Z}$, the map G_n induces an isomorphism*

$$G_n:\left(\ker G_{n+1}/\ker G_n\right)\xrightarrow{\cong}\ker\psi^n_{+,n+1}\cap\ker\psi^n_{-,n+1}.$$

Proof. Suppose $x\in\ker G_{n+1}$. Then from Lemma 2.18 we know that

$$\psi^n_{\pm,n+1}\circ G_n(x)=G_{n+1}(x)=0.$$

Hence we have

$$G_n(\ker G_{n+1}))\subset\ker\psi^n_{+,n+1}\cap\ker\psi^n_{-,n+1}.$$

Clearly G_n is injective on $\ker G_{n+1}/\ker G_n$ so it suffices to show that the image is $\ker\psi^n_{+,n+1}\cap\ker\psi^n_{-,n+1}$. To do this, for any element $x\in\ker\psi^n_{+,n+1}\cap\ker\psi^n_{-,n+1}$, we know from Lemma 2.20 that

$$x\in\ker H_n=\operatorname{Im}G_n.$$

As a result, there exists $\alpha\in\mathbf{Y}$ so that

$$x=G_n(\alpha).$$

Again from Lemma 2.18 we know that

$$G_{n+1}(\alpha)=\psi^{n+1}_{+,n}\circ G_n(\alpha)=\psi^{n+1}_{+,n}(x)=0.$$

This implies that $\alpha\in\ker G_{n+1}$. □

Lemma 6.3. *For any $n\in\mathbb{Z}$, the maps $\psi^\mu_{\pm,n}$ induce isomorphisms*

$$\psi^\mu_{+,n}:\left(\operatorname{Im}\psi^{n+2}_{+,\mu}/\operatorname{Im}\psi^{n+1}_{+,\mu}\right)\xrightarrow{\cong}\ker\psi^n_{+,n+1}\cap\ker\psi^n_{-,n+1}$$

$$\psi^\mu_{-,n}:\left(\operatorname{Im}\psi^{n+2}_{-,\mu}/\operatorname{Im}\psi^{n+1}_{-,\mu}\right)\xrightarrow{\cong}\ker\psi^n_{+,n+1}\cap\ker\psi^n_{-,n+1}$$

Proof. We only prove the lemma for positive bypasses. The proof for the negative bypasses is the similar. Let $u\in\operatorname{Im}\psi^{n+2}_{+,\mu}$. By Lemma 2.6 and Lemma 2.12, we have

$$\psi^n_{+,n+1}\circ\psi^\mu_{+,n}(u)=0\text{ and }\psi^n_{-,n+1}\circ\psi^\mu_{+,n}(u)=\psi^\mu_{+,n+1}(u)=0.$$

Hence we know

$$\psi^\mu_{+,n}(\operatorname{Im}\psi^{n+2}_{+,\mu})\subset\ker\psi^n_{+,n+1}\cap\ker\psi^n_{-,n+1}.$$

Since $\ker \psi^{\mu}_{+,n} = \operatorname{Im} \psi^{n+1}_{+,\mu}$, the map $\psi^{\mu}_{+,n}$ is injective on $\operatorname{Im} \psi^{n+2}_{+,\mu} / \operatorname{Im} \psi^{n+1}_{+,\mu}$. To show it is surjective as well, pick $x \in \ker \psi^{n}_{+,n+1} \cap \ker \psi^{n}_{-,n+1}$. Note $x \in \ker \psi^{n}_{+,n+1} = \operatorname{Im} \psi^{\mu}_{+,n}$ implies that there exists $u \in \mathbf{\Gamma}_\mu$ so that $\psi^{\mu}_{+,n}(u) = x$. Lemma 2.12 then implies that

$$\psi^{\mu}_{+,n+1}(u) = \psi^{n}_{-,n+1} \circ \psi^{\mu}_{+,n}(u) = \psi^{n}_{-,n+1}(x) = 0.$$

As a result, $u \in \ker \psi^{\mu}_{+,n+1} = \operatorname{Im} \psi^{n+2}_{+,\mu}$. □

Corollary 6.4. *(1) For any $n \in \mathbb{Z}$, there is a canonical isomorphism*

$$\left(\ker G_{n+1} / \ker G_n\right) \cong \left(\operatorname{Im} \psi^{n+2}_{+,\mu} / \operatorname{Im} \psi^{n+1}_{+,\mu}\right) \cong \left(\operatorname{Im} \psi^{n+2}_{-,\mu} / \operatorname{Im} \psi^{n+1}_{-,\mu}\right).$$

(2) For large enough n_0, there exists a (noncanonical) isomorphism

$$\mathbf{Y} \cong \left(\operatorname{Im} \psi^{n_0}_{+,\mu} / \operatorname{Im} \psi^{-n_0}_{+,\mu}\right) \cong \left(\operatorname{Im} \psi^{n_0}_{-,\mu} / \operatorname{Im} \psi^{-n_0}_{-,\mu}\right)$$

Definition 6.5. For any coprime $r, s \in \mathbb{Z}$ and any grading i, define the map F^i_n as the restriction

$$F^i_n = F_n | (\mathbf{\Gamma}_n, i).$$

where F_n is the map from Lemma 2.16.

Lemma 6.6. *Suppose $n_0 \in \mathbb{Z}$ is small enough so that $F_{n_0} = 0$ (c.f. Lemma 2.19). Then for any integer $n \geqslant n_0$ and any grading i, we have*

$$\psi^{n}_{\pm,\mu}(\ker F^i_n) = \operatorname{Im}\left(\operatorname{Proj}^{i - \frac{(n-1)p-q}{2}}_{\mu} \circ \psi^{n_0}_{\pm,\mu}\right),$$

where

$$\operatorname{Proj}^{i - \frac{(n-1)p-q}{2}}_{\mu} : \mathbf{\Gamma}_\mu \to (\mathbf{\Gamma}_\mu, i - \frac{(n-1)p-q}{2})$$

is the projection.

Proof. We only prove the lemma for positive bypasses and the proof for negative bypasses is similar. First, suppose

$$u \in \operatorname{Im}\left(\operatorname{Proj}^{i - \frac{(n-1)p-q}{2}}_{\mu} \circ \psi^{n_0}_{\pm,\mu}\right) = \operatorname{Im} \psi^{n_0}_{\pm,\mu} \cap (\mathbf{\Gamma}_\mu, i - \frac{(n-1)p-q}{2}).$$

Pick $x \in (\mathbf{\Gamma}_{n_0}, i - \frac{(n-n_0)p}{2})$ so that

$$\psi^{n_0}_{+,\mu}(x) = u.$$

Take $y = \Psi^{n_0}_{-,n}(x)$, we know from Lemma 2.6 that $y \in (\mathbf{\Gamma}_n, i)$, from Lemma 2.18 that $F_n(y) = F_{n_0}(x) = 0$, and from Lemma 2.12 that $\psi^{n}_{+,\mu}(y) = u$. As a result, we conclude $u \in \psi^{n}_{+,\mu}(\ker F^i_n)$.

Second, suppose $u \in \psi^{n}_{+,\mu}(\ker F^i_n)$ is nonzero. Pick $x_1 \in \ker F^i_n$ so that

$$\psi^{n}_{+,\mu}(x_1) = u.$$

By Lemma 2.5, Lemma 2.6, the fact that $\psi^{n}_{+,\mu}(x_1) = u \neq 0$ implies that

$$-g - \frac{p-1}{2} \leqslant i - \frac{(n-1)p-q}{2} \leqslant g + \frac{p-1}{2}.$$

Pick a large enough integer k and then take

$$x_2 = \Psi^{n}_{-,n+k}(x_1) \text{ and } x_3 = \Psi^{n+k}_{+,2n+2k-n_0}(x_2).$$

By Lemma 2.18 we have

$$F_{2n+2k-n_0}(x_3)=F_{n+k}(x_2)=F_n(x_1)=0.$$

Note that the grading j of x_3 equals to

$$j=i+\frac{kp}{2}-\frac{(n+k-n_0)p}{2}=i-\frac{(n-n_0)p}{2}$$

Hence

$$g-\frac{(2n+2k-n_0)p-q-1}{2}\leqslant j\leqslant -g+\frac{(2n+2k-n_0)p-q-1}{2}$$

since k is large. From Lemma 2.19 and the assumption on i we know that $F_{2n+2k-n_0}$ is injective on the grading j. Hence $x_3=0$. Then the following Lemma 6.7 applies to $(x,y)=(x_2,0)$ and there exists $x_4\in\mathbf{\Gamma}_{n_0}$ so that

$$\Psi^{n_0}_{-,n+k}(x_4)=x_2.$$

Thus by Lemma 2.12,

$$u=\psi^n_{+,\mu}(x_1)=\psi^{n+k}_{+,\mu}(x_2)=\psi^{n_0}_{+,\mu}(x_4)\in\operatorname{Im}\left(\operatorname{Proj}_\mu^{i-\frac{(n-1)p-q}{2}}\circ\psi^{n_0}_{+,\mu}\right).$$

□

Lemma 6.7. *Suppose $n\in\mathbb{Z}$ and $k_1,k_2\in\mathbb{N}_+$. Suppose $x\in\mathbf{\Gamma}_{n+k_1},y\in\mathbf{\Gamma}_{n+k_2}$ such that*

$$\Psi^{n+k_1}_{+,n+k_1+k_2}(x)=\Psi^{n+k_2}_{-,n+k_1+k_2}(y)$$

Then there exists $z\in\mathbf{\Gamma}_n$ so that

$$\Psi^n_{-,n+k_1}(z)=x\text{ and }\Psi^n_{+,n+k_2}(z)=y.$$

Proof. This is a restatement of Remark 5.2. The proof is similar to that of Proposition 5.1. □

6.2. **Tau invariants in a general 3-manifold.**

Definition 6.8. For any integer n and grading i, we say an element in $\operatorname{Im}F_n^i$ (*c.f.* Definition 6.5) a **homogeneous element**. For a homogeneous element $\alpha\in\mathbf{Y}$, we pick a large enough n_0 and define

$$\tau^+(\alpha):=\max_i\{i\mid\exists x\in(\mathbf{\Gamma}_{n_0},i),\ F_{n_0}(x)=\alpha\}-\frac{(n_0-1)p-q}{2}$$

$$\tau^-(\alpha):=\min_i\{i\mid\exists x\in(\mathbf{\Gamma}_{n_0},i),\ F_{n_0}(x)=\alpha\}+\frac{(n_0-1)p-q}{2}$$

$$\tau(\alpha):=1+\frac{\tau^-(\alpha)-\tau^+(\alpha)+q}{p}=\frac{\min-\max}{p}+n_0.$$

Remark 6.9. Here we fix the knot $K\subset Y$ and define the tau invariants for a homogeneous element $\alpha\in I^\sharp(Y)$. The reason why we go in this order is because (1) currently the definition of homogeneous elements depends on the choice of the knot and (2) in this paper we only focus on the Dehn surgeries of a fixed knot.

Remark 6.10. The normalization $\mp\frac{(n_0-1)p-q}{2}$ comes from the grading shifts of $\psi^{n_0}_{\pm,\mu}$ in Lemma 2.6. When K is a knot inside $Y=S^3$, we have that $\tau^\pm(\alpha)$ equal to the tau invariant $\tau_I(K)$ defined in [GLW19], where α is the unique generator of $I^\sharp(-S^3)\cong\mathbb{C}$ up to a scalar. Then $\tau(\alpha)=1-2\tau_I(K)$.

Lemma 6.11. *We have the following properties.*

(1) *Suppose n_1, n_2 are two integers and i_1, i_2 are two gradings so that there exist $x_1 \in (\mathbf{\Gamma}_{n_1}, i_1)$ and $x_2 \in (\mathbf{\Gamma}_{n_2}, i_2)$ with*
$$F_{n_1}(x_1) = F_{n_2}(x_2) \neq 0.$$
Then there exists an integer N so that
$$i_2 = i_1 - \frac{(n_2 - n_1)p}{2} + Np$$
i.e. when we send x_1 and x_2 into the same $\mathbf{\Gamma}_{n_3}$ with $n_3 > n_1, n_2$ by bypass maps, then the difference of the expected gradings of the images is divisible by p (the grading shifts of the bypass maps $\psi^n_{\pm, n+1}$ are $\mp p/2$).

(2) *Suppose we have an integer n_1, a grading i_1, and an element $x_1 \in (\mathbf{\Gamma}_{n_1}, i_1)$. Then for any integer $n_2 \geqslant n_1$ and grading i_2 so that there exists and integer $N \in [0, n_2 - n_1]$ with*
$$i_2 = i_1 - \frac{(n_2 - n_1)p}{2} + Np,$$
there exists an element $x_2 \in (\mathbf{\Gamma}_{n_2}, i_2)$ so that
$$F_{n_1}(x_1) = F_{n_2}(x_2).$$

(3) *Suppose $n \in \mathbb{Z}$ and for $1 \leqslant j \leqslant l$ we have a grading i_j and an element $x_j \in (\mathbf{\Gamma}_n, i_j)$ so that $F_n(x_1), \ldots, F_n(x_l)$ are linearly independent. Then the element*
$$\alpha = \sum_{j=1}^{l} F_n(x_i)$$
is homogeneous if and only if for any $1 \leqslant j \leqslant l$, we have
$$i_j \equiv i_1 \pmod{p}$$

Proof. (1). Take n_0 a large enough integer. For $j = 1, 2$, take $i_j \in (-\frac{p}{2}, \frac{p}{2}]$ to be the unique grading so that there exists an integer N_j with
$$i'_j = i_j - \frac{(n_0 - n_j)p}{2} + N_j p.$$
Take
$$x'_j = \Psi^{n_j + N_j}_{-, n_0} \circ \Psi^{n_j}_{+, n_j + N_j}(x_j).$$
From Lemma 2.18 we know that
$$x_j \in (\mathbf{\Gamma}_{n_0}, i'_j) \text{ and } F_{n_0}(x'_1) = F_{n_1}(x_1) = F_{n_2}(x_2) = F_{n_0}(x'_2).$$
By Lemma 2.19, we know that $x'_1 = x'_2$ and in particular, $i'_1 = i'_2$. As a result, we can take $N = N_1 - N_2$ then it is straightforward to verify that
$$i_2 = i_1 - \frac{(n_2 - n_1)p}{2} + Np.$$

(2). We can take
$$x_2 = \Psi^{n_1 + N}_{-, n_2} \circ \Psi^{n_1}_{+, n_1 + N}(x_1)$$
Then it follows from Lemma 2.6 that $x_2 \in (\mathbf{\Gamma}_{n_2}, i_2)$ and follows from Lemma 2.18 that
$$F_{n_2}(x_2) = F_{n_1}(x_1).$$

(3). The proof is similar to that of (1). □

Lemma 6.12. *For a homogeneous element α, we have the following.*

(1) $\tau^{\pm}(\alpha)$ and hence $\tau(\alpha)$ are well-defined. (i.e. they are independent of the choice of the large integer n_0.)
(2) We have $\tau(\alpha) \in \mathbb{Z}$.
(3) For any integer n and grading i, the following two statements are equivalent.
(a) There exists $x \in (\mathbf{\Gamma}_n, i)$ so that $F_n(x) = \alpha$.
(b) We have $n \geqslant \tau(\alpha)$ and there exists $N \in \mathbb{Z}$ so that $N \in [0, n - \tau(\alpha)]$

$$i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (n - \tau(\alpha))p}{2} + Np.$$

(4) We have

$$\tau^+(\alpha) \geqslant -\frac{p-1}{2} - g \text{ and } \tau^-(\alpha) \leqslant \frac{p-1}{2} + g.$$

Proof. (1). Suppose α is a homogeneous element. Then by definition there exists $x \in (\mathbf{\Gamma}_n, i)$ for some integer n and grading i so that

$$F_n(x) = \alpha.$$

Then for large enough n_0, we can take

$$y = \psi^n_{+,n_0}(x)$$

and from Lemma 2.18 implies that

$$F_{n_0}(y) = \alpha$$

and hence $\tau^{\pm}(\alpha)$ exists.

To show the value of $\tau^{\pm}(\alpha)$ is independent of n_0 as long as it is large enough, a combination of Lemma 2.5 and Lemma 2.6 implies that the map

$$\psi^{n_0}_{-,n_0+1} : (\mathbf{\Gamma}_{n_0}, i) \to (\mathbf{\Gamma}_{n_0+1}, i + \frac{p}{2})$$

is an isomorphism for any $i > g - \frac{n_0p-q-1}{2}$. Then Lemma 2.18 implies that τ^+ is well-defined. The argument for τ^- is similar.

(2). It follows directly from Lemma 6.11 part (1). (3). We first prove that (b) implies (a). First, suppose $n \in \mathbb{Z}$ is large enough and

$$x_{\pm} \in (\mathbf{\Gamma}_n, \tau^{\pm}(\alpha) \pm \frac{(n-1)p+q}{2})$$

so that $F_n(x_{\pm}) = \alpha$. Let

$$x'_{\pm} = \Psi^n_{\pm,2n-\tau(\alpha)}(x_{\pm}).$$

It follows from Lemma 2.6 that

$$x'_{\pm} \in (\mathbf{\Gamma}_{2n-\tau(\alpha)}, \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2}).$$

From Lemma 2.18 we know that

$$F_{2n-\tau(\alpha)}(x'_+) = \alpha = F_{2n-\tau(\alpha)}(x'_-).$$

By Lemma 2.19 this implies that

$$x'_+ = x'_-.$$

Hence Lemma 6.7 applies and there exists $z \in \mathbf{\Gamma}_{\tau(\alpha)}$ so that

$$\Psi^{\tau(\alpha)}_{\mp,n}(z) = x_{\pm}.$$

Again Lemma 2.6 implies that z is in the grading

$$z \in (\mathbf{\Gamma}_{\tau(\alpha)}, \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2})$$

and and Lemma 2.18 implies

$$F_{\tau(\alpha)}(z) = \alpha.$$

Then Lemma 6.11 part (2) implies that (b) $\Rightarrow$ (a).

To show that (a) implies (b). Suppose there exists $x \in (\mathbf{\Gamma}_n, i)$ so that $F_n(x) = \alpha$. From the above argument, we already known that there exists

$$z \in (\mathbf{\Gamma}_{\tau(\alpha)}, \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2})$$

so that

$$F_{\tau(\alpha)}(z) = \alpha$$

Hence Lemma 6.11 part (1) applies and we know that there exists $N \in \mathbb{Z}$ so that

$$i = \frac{\tau^+(\alpha)+\tau^-(\alpha)-(n-\tau(\alpha))p}{2} + Np.$$

If $N > n - \tau(\alpha)$, we can take a large enough n_0 and

$$x' = \Psi^n_{-,n_0}(x).$$

It follows from Lemma 2.6 that

$$x' \in (\mathbf{\Gamma}_{n_0}, i') \text{ with } i' > \tau^+(\alpha) + \frac{(n-1)p-q}{2}.$$

Then Lemma 2.18 implies that

$$F_{n_0}(x') = \alpha$$

which contradicts to the definition of τ^+ in Definition 6.8. Similarly if $N < 0$ we can take

$$x' = \Psi^n_{+,n_0}(x)$$

which would be an element contradicting the definition of τ^-. When $n < \tau(\alpha)$ we have $n-\tau(\alpha) < 0$ so there is always a contradiction.

(4). It follows from the definition of $\tau^{\pm}$ and Lemma 2.19 that F_n is an isomorphism when restricted to the direct sum of p consecutive middle gradings of $\mathbf{\Gamma}_n$ when n is large. ☐

Lemma 6.13. *For any $n \in \mathbb{Z}$ we have that*

$$\operatorname{Im} F_n = \operatorname{Span}\{\alpha \in \mathbf{Y} \mid \alpha \text{ homogeneous and } \tau(\alpha) \leqslant n\}$$

Proof. Suppose $\alpha \in \operatorname{Im} F_n$. Let

$$\alpha = \sum_i \alpha_i \text{ where } \alpha_i \in \operatorname{Im} F_n^i \text{ is homogeneous.}$$

From Lemma 6.12 we know that $\tau(\alpha_i) \leqslant n$ for all i. On the other hand, suppose

$$\alpha = \sum_i \alpha_i \text{ where } \tau(\alpha_i) \leqslant n \text{ for all } i.$$

By Lemma 6.12 part (3) we can pick $z_i \in \mathbf{\Gamma}_{\tau(\alpha_i)}$ so that

$$F_{\tau(\alpha_i)}(z_i) = \alpha_i.$$

Then from Lemma 2.18 we know

$$\alpha = F_n(\sum_i \Psi_{+,n}^{\tau(\alpha_i)}(z_i)).$$

□

6.3. **A basis for framed instanton homology.** We pick a set of basis $\mathfrak{B}$ for $\mathbf{Y}$ as follows. First

$$\mathfrak{B} = \bigcup_{n\in\mathbb{Z}} \mathfrak{B}_n.$$

To construct the set $\mathfrak{B}_n$ first let $\mathfrak{B}_n = \varnothing$ if $F_n = 0$. By Lemma 2.19 this means $\mathfrak{B}_n = \varnothing$ for all small enough n. Write

$$\mathfrak{B}_{\leqslant n} = \bigcup_{k\leqslant n} \mathfrak{B}_k.$$

We pick the set $\mathfrak{B}_n$ inductively. Note we have taken $\mathfrak{B}_n = \varnothing$ for n with $F_n = 0$. Suppose we have already constructed the set $\mathfrak{B}_{\leqslant n-1}$ that consists of homogeneous elements and is a basis of $\operatorname{Im} F_{n-1}$, we pick the set $\mathfrak{B}_n$ so that $\mathfrak{B}_n$ consists of homogeneous elements with $\tau = n$, and the set

$$\mathfrak{B}_{\leqslant n} = \mathfrak{B}_{\leqslant n-1} \cup \mathfrak{B}_n$$

forms a basis of $\operatorname{Im} F_n$. Note Lemma 6.13 implies that $\mathfrak{B}_n$ exists and

$$|\mathfrak{B}_n| = \dim_{\mathbb{C}}\Big(\operatorname{Im} F_n / \operatorname{Im} F_{n-1}\Big).$$

For any $n, k \in \mathbb{Z}$ so that $k \leqslant n-2$, define maps

$$\eta_{\pm,k}^n : \mathfrak{B}_n \to \mathbf{\Gamma}_k$$

as follows: for any $\alpha \in \mathfrak{B}_n \subset \operatorname{Im} F_n$, since α is homogeneous and $\tau(\alpha) = n$, we can pick

$$z \in (\mathbf{\Gamma}_n, \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2})$$

by Lemma 6.12 part (3) so that $F_n(z) = \alpha$. Then define

$$\eta_{\pm,k}^n(\alpha) = \psi_{\pm,k}^{\mu} \circ \psi_{\pm,\mu}^n(z)$$

.

Lemma 6.14. *Suppose $n, k \in \mathbb{Z}$ so that $k \leqslant n-2$.*

(1) *The maps $\eta_{\pm,k}^n$ are all well-defined.*
(2) *We have $\eta_{+,n-2}^n = c_n \cdot \eta_{-,n-2}^n$. for some scalar $c_n \in \mathbb{C}\backslash\{0\}$.*
(3) *Elements in $\operatorname{Im} \eta_{\pm,k}^n \subset \mathbf{\Gamma}_k$ are linearly independent.*
(4) *$\operatorname{Im} \eta_{\pm,n-2}^n$ forms a basis for $\ker \psi_{+,n-1}^{n-2} \cap \ker \psi_{-,n-1}^{n-2}$.*
(5) *For any $\alpha \in \mathfrak{B}_n$ we have*

$$\eta_{\pm,k}^n(\alpha) \in (\mathbf{\Gamma}_k, \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2} \mp \frac{(n-2-k)p}{2}).$$

(6) *We have*

$$\psi_{\mp,k}^{k-1} \circ \eta_{\pm,k-1}^n = \eta_{\pm,k}^n, \text{ and } \psi_{\pm,k}^{k-1} \circ \eta_{\pm,k-1}^n = 0.$$

Proof. (1). We only work with $\eta^n_{+,k}$ and the arguments for $\eta^n_{-,k}$ are similar. Suppose there are $z_1, z_2 \in (\mathbf{\Gamma}_n, i)$ so that $F_n(z_1) = F_n(z_2) = \alpha$, where $i = \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2}$. Then

$$z_1 - z_2 \in \ker F_n^i$$

and by Lemma 6.6 we have

$$\psi^n_{+,\mu}(z_1 - z_2) \in \psi^n_{+,\mu}(\ker F_n^i) \subset \operatorname{Im} \psi^{n_0}_{+,\mu} \subset \operatorname{Im} \psi^{k+1}_{+,\mu}.$$

Here $n_0 \in \mathbb{Z}$ is a small enough integer. As a result,

$$\eta^n_{+,k}(\alpha) = \psi^\mu_{+,k} \circ \psi^n_{+,\mu}(z_1) = \psi^\mu_{+,k} \circ \psi^n_{+,\mu}(z_2)$$

is well-defined.

(2). It follows directly from Lemma 2.11. Note that in Section 2.3 we do not fix the scalars of the second commutative diagram of Lemma 2.11.

(3). We only work with $\eta^n_{+,k}$ and the arguments for $\eta^n_{-,k}$ are similar. Suppose

$$\mathfrak{B}_n = \{\alpha_1, \ldots, \alpha_l\}, \text{ where } l = |\mathfrak{B}_n| = \dim_{\mathbb{C}} \bigg(\operatorname{Im} F_n / \operatorname{Im} F_{n-1} \bigg).$$

Suppose there exists $\lambda_1, \ldots, \lambda_l$ so that

$$\sum_{j=1}^{l} \lambda_j \cdot \eta^n_{+,k}(\alpha_j) = 0.$$

Pick $z_j \in (\mathbf{\Gamma}_n, \frac{\tau^+(\alpha_j)+\tau^-(\alpha_j)}{2})$ so that $F_n(z_j) = \alpha_i$, we have

$$\psi^\mu_{+,k} \circ \psi^n_{+,\mu} \bigg(\sum_{j=1}^{l} \lambda_j z_j \bigg) = 0.$$

As a result, there exists $x \in \mathbf{\Gamma}_{n-1}$ so that

$$\psi^{k+1}_{+,\mu}(x) = \psi^n_{+,\mu} \bigg(\sum_{j=1}^{l} \lambda_j z_j \bigg),$$

Note from Lemma 2.12 we know

$$\psi^n_{+,\mu} \circ \Psi^{k+1}_{-,n}(x) = \psi^{k+1}_{+,\mu}(x) = \psi^n_{+,\mu} \bigg(\sum_{j=1}^{l} \lambda_j z_j \bigg)$$

so as a result there exists $y \in \mathbf{\Gamma}_{n-1}$ so that

$$\sum_{j=1}^{l} \lambda_j z_j = \Psi^{k+1}_{-,n}(x) + \psi^{n-1}_{+,n}(y).$$

Hence by Lemma 2.18 we have

$$\begin{aligned} \sum_{j=1}^{l} \lambda_j \alpha_j &= F_n(\sum_{j=1}^{l} \lambda_j z_j) \\ &= F_n \circ \Psi^{k+1}_{-,n}(x) + F_n \circ \psi^{n-1}_{+,n}(y) \\ &= F_k(x) + F_{n-1}(y) \\ &\subset \operatorname{Im} F_{n-1}. \end{aligned}$$

Since α_j form a basis of $\mathfrak{B}_n$, the sum cannot be in $\operatorname{Im} F_{n-1}$ except $\lambda_i = 0$ for all i.

(4). For $\alpha \in \mathfrak{B}_n$, pick $z \in (\mathbf{\Gamma}_n, \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2})$ so that $F_n(z) = \alpha$. Then by definition

$$\eta^n_{+,n-2}(\alpha) = \psi^\mu_{+,n-2} \circ \psi^n_{+,\mu}(z).$$

Now we can compute

$$\psi^{n-2}_{+,n-1} \circ \eta^n_{+,n-2}(\alpha) = \psi^{n-2}_{+,n-1} \circ \psi^\mu_{+,n-2} \circ \psi^n_{+,\mu}(z) = 0,$$

and by Lemma 2.12

$$\psi^{n-2}_{-,n-1} \circ \eta^n_{+,n-2}(\alpha) = \psi^{n-2}_{-,n-1} \circ \psi^\mu_{+,n-2} \circ \psi^n_{+,\mu}(z) = \psi^\mu_{+,n-1} \circ \psi^n_{+,\mu}(z) = 0.$$

Hence

$$\eta^n_{+,n-2}(\alpha) \in \ker \psi^{n-2}_{+,n-1} \cap \ker \psi^{n-2}_{-,n-1}.$$

Then (4) follows from (3), Lemma 6.2, and $\operatorname{Im} F_n = \ker G_{n-1}$.

(5). It follows directly from the construction of $\eta^n_{\pm,k}$ and Lemma 2.6.

(6). It follows from the construction of $\eta^n_{\pm,k}$, the commutativity in Lemma 2.12 and the exactness in Lemma 2.6. □

Convention. We can define

$$\tilde{\eta}^n_{+,k} = \eta^n_{+,k} \text{ and } \tilde{\eta}^n_{-,k} = c_n \cdot \eta^n_{-,k}$$

so that

$$\tilde{\eta}^n_{+,n-2} = \tilde{\eta}^n_{-,n-2}$$

and the new maps satisfy all properties in Lemma 6.14 except (2). We will use $\eta^n_{+,k}$ to denote $\tilde{\eta}^n_{+,k}$ in latter sections.

7. The map in the third exact triangle

In this section, we construct the map l in Proposition 3.9 and Proposition 3.12 show it satisfies the exactness and the commutative diagram. We still adapt conventions in Section 2.3. We restate the propositions as follows and no longer use the notations l, l' for maps.

Proposition 7.1. *Suppose $n \in \mathbb{Z}$ is fixed and $k \in \mathbb{Z}$ is large enough. Then There is an exact triangle*

$$\begin{tikzcd} \mathbf{\Gamma}_n \arrow[rr, "\Phi^n_{n+k}"] & & \mathbf{\Gamma}_{n+k} \oplus \mathbf{\Gamma}_{n+k} \arrow[dl, "\Phi^{n+k}_{n+2k}"] \\ & \mathbf{\Gamma}_{n+2k}. \arrow[ul, "\Phi^{n+2k}_n"] & \end{tikzcd}$$

where two of the maps are already constructed

$$\Phi^n_{n+k} := (\Psi^n_{-,n+k}, \Psi^n_{+,n+k}) : \mathbf{\Gamma}_n \to \mathbf{\Gamma}_{n+k} \oplus \mathbf{\Gamma}_{n+k}$$

$$\Phi^{n+k}_{n+2k} := \Psi^{n+k}_{+,n+2k} - \Psi^{n+k}_{+,n+2k} : \mathbf{\Gamma}_{n+k} \oplus \mathbf{\Gamma}_{n+k} \to \mathbf{\Gamma}_{n+2k}.$$

Proposition 7.2. *Suppose $n \in \mathbb{Z}$ is fixed and $k \in \mathbb{Z}$ is large enough. Suppose Φ^{n+2k-1}_{n-1} is constructed in Proposition 7.1. Then there are two commutative diagrams up to scalars.*

$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n+2k+1}{2}} \arrow[r, "\psi^{n+k+1}_{+,\mu}\circ\psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"] \arrow[d, "\Psi^{n+k+1}_{+,n+2k}\circ\psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"] & \mathbf{\Gamma}_{\mu} \arrow[d, "\psi^{\mu}_{+,n}"] \\
\mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_{n}"] & \mathbf{\Gamma}_{n}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n+2k+1}{2}} \arrow[r, "\psi^{n+k+1}_{-,\mu}\circ\psi^{\frac{2n+2k}{2}}_{+,n+k+1}"] \arrow[d, "\Psi^{n+k+1}_{-,n+2k+1}\circ\psi^{\frac{2n+2k+1}{2}}_{+,n+k+1}"] & \mathbf{\Gamma}_{\mu} \arrow[d, "\psi^{\mu}_{-,n}"] \\
\mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_{n}"] & \mathbf{\Gamma}_{n}
\end{tikzcd}$$

7.1. **Characterizations of the kernel and the image.** Before constructing Φ^{n+2k}_n, we characterize the spaces $\ker\Phi^n_{n+k}$ and $\operatorname{Im}\Phi^{n+k}_{n+2k}$. These results will motivate the construction of Φ^{n+2k}_n to make

$$\operatorname{Im}\Phi^{n+2k}_n = \ker\Phi^n_{n+k} \text{ and } \ker\Phi^{n+2k}_n = \operatorname{Im}\Phi^{n+k}_{n+2k}.$$

Since Φ^n_{n+k} and Φ^{n+k}_{n+2k} are constructed by bypass maps, it suffices to consider their restrictions on each grading.

Lemma 7.3. *Suppose $n\in\mathbb{Z}$ is fixed and $k\in\mathbb{Z}$ is large enough. Let*

$$\mathrm{Proj}^i_n : \mathbf{\Gamma}_n \to (\mathbf{\Gamma}_n, i)$$

be the projection. Then we have

$$\ker\Phi^n_{n+k}\cap(\mathbf{\Gamma}_n, i) = \operatorname{Im}\left(\mathrm{Proj}^i_n\circ G_n\right).$$

Proof. We will need to apply Lemma 2.20. Following conventions in Section 2.3, we have

$$H_n = \psi^n_{+,n+1} - \psi^n_{-,n+1}. \tag{7.1}$$

Suppose $x\in\operatorname{Im}\left(\mathrm{Proj}^i_n\circ G_n\right)$. Pick $\alpha\in\mathbf{Y}$ and $y\in\mathbf{\Gamma}_n$ so that

$$G_n(\alpha) = x+y \text{ and no homogeneous part of } y \text{ is in grading } i.$$

When k is large enough we know from Lemma 2.19 that

$$G_{n+k}\equiv 0.$$

In particular, from Lemma 2.18

$$\psi^n_{\pm,n+k}(x) + \psi^n_{\pm,n+k}(y) = G_{n+k}(\alpha) = 0.$$

Since $\psi^n_{\pm,n+k}$ are homogeneous, we know that

$$\psi^n_{\pm,n+k}(x) = 0$$

which implies that $x\in\ker\Phi^n_{n+k}\cap(\mathbf{\Gamma}_n, i)$.

Next, suppose $x\in\ker\Phi^n_{n+k}\cap(\mathbf{\Gamma}_n, i)$. We take $x^i_n = x$ and we will pick $x^j_n\in(\mathbf{\Gamma}_n, j)$ for all $j\neq i$ so that

$$\sum_j x^j_n \in \ker H_n = \operatorname{Im} G_n.$$

We will use the notation x^b_a to denote an element in $(\mathbf{\Gamma}_a, b)$. Recall that from Lemma 2.6, the grading shifts of $\psi^n_{\pm,n+1}$ are $\mp\frac{p}{2}$. Take

$$x^{i+\frac{(k-1)p}{2}}_{n+k-1} = \Psi^n_{-,n+k-1}(x) \text{ and } x^{i+\frac{(k+1)p}{2}}_{n+k-1} = 0.$$

Since $x\in\ker\Phi^n_{n+k}\cap(\mathbf{\Gamma}_n,i)$ we know that

$$\psi^{n+k-1}_{-,n+k}(x^{i+\frac{(k-1)p}{2}}_{n+k-1})=0=\psi^{n+k-1}_{+,n+k}(x^{i+\frac{(k+1)p}{2}}_{n+k-1}).$$

Hence from Lemma 6.7, there exists

$$x^{i+\frac{kp}{2}}_{n+k-2}\in(\mathbf{\Gamma}_{n+k-2},i+\frac{kp}{2})$$

so that

$$\psi^{n+k-2}_{-,n+k-1}(x^{i+\frac{kp}{2}}_{n+k-2})=x^{i+\frac{(k+1)p}{2}}_{n+k-1}=0\text{ and }\psi^{n+k-2}_{+,n+k-1}(x^{i+\frac{kp}{2}}_{n+k-2})=x^{i+\frac{(k-1)p}{2}}_{n+k-1}.$$

Then we can take

$$x^{i+\frac{(k+2)p}{2}}_{n+k-2}=0\text{ and }x^{i+\frac{(k-2)p}{2}}_{n+k-2}=\Psi^n_{-,n+k-2}(x).$$

We can apply the same argument and use Lemma 6.7 to find

$$x^{i+\frac{(k+3)p}{2}}_{n+k-3},\ x^{i+\frac{(k+1)p}{2}}_{n+k-3},\ x^{i+\frac{(k-1)p}{2}}_{n+k-3},\ x^{i+\frac{(k-3)p}{2}}_{n+k-3}\in\mathbf{\Gamma}_{n+k-3}$$

so that $\psi^{n+k-3}_{\pm,n+k-2}$ send them to corresponding elements in $\mathbf{\Gamma}_{n+k-2}$. Repeating this argument, we can obtain elements

$$x^{i+pj}_n\in(\mathbf{\Gamma}_n,i+pj)\text{ for }j\in[-k,k]\cap\mathbb{Z}$$

so that $x^i_n=x$, $\psi^n_{-,n+1}(x^{i+pk}_n)=0$, $\psi^n_{+,n+1}(x^{i-pk}_n)=0$, and for any $j\in[-k,k-1]\cap\mathbb{Z}$ we have

$$\psi^n_{-,n+1}(x^{i+pj}_n)=\psi^n_{+,n+1}(x^{i+p(j+1)}_n).$$

Hence it is straightforward to check that

$$y=\sum_{j=-k}^{k}x^{i+pj}_n\in\ker(\psi^n_{+,n+1}-\psi^n_{-,n+1})=\ker H_n=\operatorname{Im}G_n.$$

$\square$

Lemma 7.4. *Suppose $\alpha\in\mathbf{Y}$ is a homogeneous element and*

$$\alpha=\sum_{j=1}^{l}\lambda_j\cdot\alpha_j$$

where $\lambda_j\neq0$ and $\alpha_j\in\mathfrak{B}$ for $1\leqslant j\leqslant l$. Let n be an integer, i be a grading and k be a large enough integer. For an element $x\in(\mathbf{\Gamma}_{n+2k},i)$ so that $F_{n+2k}(x)=\alpha$, the following is true.

(1) We have

$$\tau^+(\alpha)=\min_{1\leqslant j\leqslant l}\{\tau^+(\alpha_j)\}\text{ and }\tau^-(\alpha)=\max_{1\leqslant j\leqslant l}\{\tau^-(\alpha_j)\}.$$

(2) We have $x\in\operatorname{Im}\Phi^{n+k}_{n+2k}$ if and only if for any $1\leqslant j\leqslant l$, at least one of the following inequalities holds

$$i\geqslant\tau^-(\alpha_j)-\frac{(n-1)p-q}{2}\text{ and }i\leqslant\tau^+(\alpha_j)+\frac{(n-1)p-q}{2}.$$

(3) If $x\notin\operatorname{Im}\Phi^{n+k}_{n+2k}$ then there exists $j,N\in\mathbb{Z}$ so that $1\leqslant j\leqslant l$, $0\leqslant N\leqslant\tau(\alpha_j)-n-2$, and

$$i=\tau^+(\alpha_j)+\frac{(n-1)p-q}{2}+(N+1)p.$$

Proof. (1). We only prove the result for τ^+ and the argument for τ^- is similar. We make the following two claims first.

Claim 1. For any homogeneous element (not necessarily elements in $\mathfrak{B}$) α_1 and α_2 so that $\alpha_1+\alpha_2$ is also homogeneous, if $\tau^+(\alpha_1)>\tau^+(\alpha_2)$ then $\tau^+(\alpha_1+\alpha_2)=\tau^+(\alpha_2)$.

To prove Claim 1, let n be large enough. From Lemma 6.11 part (3) we know that

$$\tau^+(\alpha_1)\equiv\tau^+(\alpha_2)\equiv\tau^+(\alpha_1+\alpha_2)\ (\mathrm{mod}\ p).$$

Assuming $\tau^+(\alpha_1+\alpha_2)>\tau^+(\alpha_2)$. Let

$$\tau^+=\min\{\tau^+(\alpha_1),\tau^+(\alpha_1+\alpha_2)\}>\tau^+(\alpha_2).$$

From Lemma 6.12 part (3) and (4) we then know that there exist

$$x_1,x_3\in(\mathbf{\Gamma}_n,\tau^++\frac{(n-1)p-q}{2})$$

so that

$$F_n(x_1)=\alpha_1 \text{ and } F_n(x_3)=\alpha_1+\alpha_2.$$

As a result,

$$F_n(x_3-x_1)=\alpha_2$$

which contradicts the definition of $\tau^+(\alpha_2)$.

Claim 2. Suppose $\alpha_1,\dots,\alpha_u\in\mathfrak{B}$ are pair-wise distinct elements in $\mathfrak{B}$ so that

$$\tau^+(\alpha_1)=\tau^+(\alpha_2)=\cdots=\tau^+(\alpha_u)=\tau^+.$$

Suppose

$$\alpha'=\sum_{i=1}^{u}\lambda_j\cdot\alpha_j$$

and suppose it is homogeneous. Then $\tau^+(\alpha')=\tau^+$.

To prove Claim 2, assume that $\tau^+(\alpha)>\tau^+$. Without loss of generality, assume that $\lambda_1\neq 0$ and

$$\tau^-(\alpha_1)=\min_{1\leqslant j\leqslant k}\{\tau^-(\alpha_j)\}.$$

Then a similar argument as in the proof of Claim 1 implies that

$$\tau^-(\alpha)\leqslant\tau^-(\alpha_1).$$

Note we assumed that $\tau^+(\alpha)>\tau^+=\tau^+(\alpha_1)$. Hence by Definition 6.8, $\tau(\alpha)<\tau(\alpha_1)$, which contradicts the construction of the set $\mathfrak{B}$.

Now we prove part (1). Suppose $\alpha_1,\dots,\alpha_l\in\mathfrak{B}$ are pair-wise distinct elements in $\mathfrak{B}$. Let

$$\alpha=\sum_{j=1}^{l}\lambda_j\cdot\alpha_j.$$

We want to show that

$$\tau^+(\alpha)=\min_{1\leqslant j\leqslant l}\{\tau^+(\alpha_j)\}.$$

To do this, relabel the elements α_j if necessary so that

$$\tau^+(\alpha_1)=\tau^+(\alpha_2)=\cdots=\tau^+(\alpha_u)<\tau^+(\alpha_{u+1})\leqslant\tau^+(\alpha_{u+2})\leqslant\cdots\leqslant\tau^+(\alpha_l).$$

Since α is homogeneous, from Lemma 6.11 part (3), we know that the sum

$$\sum_{j=1}^{v} \lambda_j \cdot \alpha_j$$

is also homogeneous for any $v = 1, \dots, l$. Applying Claim 2, we conclude that

$$\tau^+\left(\sum_{j=1}^{u} \lambda_j \cdot \alpha_j\right) = \tau^+(\alpha_1).$$

Hence we can apply Claim 1 repeatedly to conclude that

$$\tau^+\left(\sum_{j=1}^{l} \lambda_j \cdot \alpha_j\right) = \tau^+(\alpha_1) = \min_{1 \leqslant j \leqslant l}\{\tau^+(\alpha_j)\}.$$

(2). If $x \in \operatorname{Im}\Phi^{n+k}_{n+2k}$, then there exists $y \in (\mathbf{\Gamma}_{n+k}, i - \frac{kp}{2})$ and $z \in (\mathbf{\Gamma}_{n+k}, i + \frac{kp}{2})$ so that

$$x = \Psi^{n+k}_{-,n+2k}(y) + \Psi^{n+k}_{+,n+2k}(z).$$

By assumption

$$F_{n+2k}(x) = \alpha = \sum_{j=1}^{l} \lambda_j \cdot \alpha_j$$

with $\lambda_j \neq 0$ and α homogeneous. By Lemma 2.18 we have

$$\alpha = F_{n+k}(y + z).$$

Since $\mathfrak{B}$ forms a basis for $\mathbf{Y}$, we can write

$$F_{n+k}(y) = \sum_{j=1}^{L} \lambda'_j \cdot \alpha_j \text{ and } F_{n+k}(z) = \sum_{j=1}^{L} \lambda''_j \cdot \alpha_j,$$

where $L = |\mathfrak{B}|$. Then for any $1 \leqslant j \leqslant l$, at least one of λ'_j and λ''_j is nonzero. Since both $F_{n+k}(y)$ and $F_{n+k}(z)$ are homogeneous, from part (1) we know

$$i - \frac{kp}{2} \geqslant \tau^-(\alpha_j) - \frac{(n+k-1)p - q}{2} \text{ when } \lambda'_j \neq 0$$

$$\text{and } i + \frac{kp}{2} \leqslant \tau^+(\alpha_j) + \frac{(n+k-1)p - q}{2} \text{ when } \lambda''_j \neq 0.$$

Conversely, suppose for any $1 \leqslant j \leqslant l$ at least one of the following inequalities holds

$$i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p - q}{2} \text{ and } i \leqslant \tau^+(\alpha_j) + \frac{(n-1)p - q}{2}.$$

We show $x \in \operatorname{Im}\Phi^{n+k}_{n+2k}$. If $|i| \geqslant \frac{(n+2k)p-q-1}{2} - g$, then from Corollary 2.9, clearly either $\Psi^{n+k}_{+,n+2k}$ or $\Psi^{n+k}_{-,n+2k}$ is surjective onto $(\mathbf{\Gamma}_{n+2k}, i)$ and then $x \in \operatorname{Im}\Phi^{n+k}_{n+2k}$.

Then we assume $|i| < \frac{np-q-1}{2} - g$. By Lemma 2.19, the map F_{n+2k} is injective when restricted to $(\mathbf{\Gamma}_{n+2k}, i)$. From the definition of $\tau^{\pm}$ in 6.8, for any j such that

$$i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p - q}{2},$$

we can then pick $y_j \in (\mathbf{\Gamma}_{n+k}, i - \frac{kp}{2})$ so that

$$F_{n+k}(y_j) = \lambda_j \cdot \alpha_j$$

and let y be the sum of all such y_j. Similarly, for any other j with $1 \leqslant j \leqslant l$, we have

$$i \leqslant \tau^+(\alpha_j) + \frac{(n-1)p-q}{2}$$

we can take $z_j \in (\mathbf{\Gamma}_{n+k}, i + \frac{kp}{2})$ so that

$$F_{n+k}(z_j) = \lambda_j \cdot \alpha_j$$

and let z be the sum of all such z_j. Then from Lemma 2.18 it is straightforward to check that

$$F_{n+2k}(\Psi^{n+k}_{-,n+2k}(y) + \Psi^{n+k}_{+,n+2k}(z)) = \alpha = F_{n+2k}(x).$$

Since we have assumed that F_{n+2k} is injective, we conclude that

$$x = \Psi^{n+k}_{-,n+2k}(y) + \Psi^{n+k}_{+,n+2k}(z) \in \operatorname{Im} \Phi^{n+k}_{n+2k}.$$

(3). If $x \notin \operatorname{Im} \Phi^{n+k}_{n+2k}$, then part (2) means that there exists some j so that

$$\tau^+(\alpha_j) + \frac{(n-1)p-q}{2} < i < \tau^-(\alpha_j) - \frac{(n-1)p-q}{2}.$$

Note by Lemma 6.11 part (3), we must have

$$i \equiv \tau^-(\alpha_j) - \frac{(n-1)p-q}{2} \equiv \tau^+(\alpha_j) + \frac{(n-1)p-q}{2} \pmod{p}.$$

By direct calculation, we have

$$(\tau^-(\alpha_j) - \frac{(n-1)p-q}{2}) - (\tau^+(\alpha_j) + \frac{(n-1)p-q}{2}) = (\tau(\alpha_j) - n)p$$

Then we can choose N with $0 \leqslant N \leqslant \tau(\alpha_j) - n - 2$ as desired. $\square$

7.2. **The construction of the map.** Since Φ^n_{n+k} and Φ^{n+k}_{n+2k} are homogeneous, we can construct Φ^{n+2k}_n on each grading to achieve the exactness and the commutativity. From the grading shifts in Lemma 2.6 and Lemma 2.13, the map Φ^{n+2k}_n should be grading preserving. from Lemma 2.5, for any grading i with

$$|i| > g + \frac{|np-q|-1}{2},$$

we have $(\mathbf{\Gamma}_n, i) = 0$. From Corollary 2.9, we know either $\Psi^{n+k}_{+,n+2k}$ or $\Psi^{n+k}_{-,n+2k}$ is surjective onto $(\mathbf{\Gamma}_{n+2k}, i)$ for such grading i. Thus, on such grading i, the zero map satisfies the exactness for Φ^{n+2k}_n (though we still have to verify the commutativity in Proposition 7.2).

On the other hand , from Lemma 2.19, the restriction of F_{n+2k} on consecutive p middle gradings is an isomorphism. In particular, when $p = 1$, it is an isomorphism when restricted to each middle grading. Also from Lemma 7.3, it seems that the definition of Φ^{n+2k}_{n+k} on $(\mathbf{\Gamma}_{n+2k}, i)$ should involve $\mathrm{Proj}^i_n \circ G_n$. However, if we only take

$$\mathrm{Proj}^i_n \circ G_n \circ F_{n+2k}$$

as the definition, current techniques are not enough to show the exactness and the commutativity.

We resolve this issue by introducing an isomorphism

$$I : \mathbf{Y} \xrightarrow{\cong} \mathbf{Y}$$

and define

$$\Phi^{n+2k}_n(x) = \mathrm{Proj}^i_n \circ G_n \circ I \circ F_{n+2k}(x) \text{ for } x \in (\mathbf{\Gamma}_{n+2k}, i). \tag{7.2}$$

The construction of I is noncanonical but it helps us to prove the exactness and commutativity.

Remark 7.5. In the first arXiv version of this paper, we deal with the special case $Y = S^3$. In this case $\mathbf{Y} \cong \mathbb{C}$ so up to a scalar we have $I = \mathrm{Id}$. In this special case indeed we could prove the exactness and commutativity without explicitly write down the isomorphism I as follows.

We first define the map I on the basis

$$\mathfrak{B} = \bigcup_{n\in\mathbb{Z}} \mathfrak{B}_n$$

of $\mathbf{Y}$ chosen in Section 6.3 that consists of homogeneous elements and then extend the map on the whole space linearly. We will show it is an isomorphism.

Fix $n_0 \in \mathbb{Z}$ small enough so that Corollary 2.9 and Lemma 2.19 apply. For any $\alpha \in \mathfrak{B}_n$, there exists a grading $i(\alpha) \in (-\frac{p}{2}, \frac{p}{2}]$ so that there exists $N(\alpha) \in \mathbb{Z}$ with

$$i(\alpha) = \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2} - \frac{(\tau(\alpha) - 2 - n_0)p}{2} + N(\alpha)p.$$

By Lemma 6.14 part (2) and (6) (and the convention after the lemma), we know that for any $\alpha \in \mathfrak{B}_n$

$$\Psi^{n_0+N(\alpha)}_{-,\tau(\alpha)-2}\left(\eta^{\tau(\alpha)}_{+,n_0+N(\alpha)}(\alpha)\right) = \eta^{\tau(\alpha)}_{+,\tau(\alpha)-2}(\alpha) = \eta^{\tau(\alpha)}_{-,\tau(\alpha)-2}(\alpha) = \Psi^{\tau(\alpha)-2-N(\alpha)}_{+,\tau(\alpha)-2}\left(\eta^{\tau(\alpha)}_{-,\tau(\alpha)-2-N(\alpha)}(\alpha)\right).$$

Then by Lemma 6.7, there exists $w \in (\mathbf{\Gamma}_{n_0}, i(\alpha))$ so that

$$\Psi^{n_0}_{+,n_0+N(\alpha)}(w) = \eta^{\tau(\alpha)}_{+,n_0+N(\alpha)}(\alpha) \text{ and } \Psi^{n_0}_{-,\tau(\alpha)-2-N(\alpha)}(w) = \eta^{\tau(\alpha)}_{-,\tau(\alpha)-2-N(\alpha)}(\alpha). \tag{7.3}$$

Let

$$\mathrm{Proj} : \mathbf{\Gamma}_{n_0} \to \bigoplus_{i\in(-\frac{p}{2},\frac{p}{2}]} (\mathbf{\Gamma}_{n_0}, i).$$

From Lemma 2.19, we know

$$\mathrm{Proj} \circ G_{n_0} : \mathbf{Y} \to \bigoplus_{i\in(-\frac{p}{2},\frac{p}{2}]} (\mathbf{\Gamma}_{n_0}, i)$$

is an isomorphism. Hence we define

$$I(\alpha) = (\mathrm{Proj} \circ G_{n_0})^{-1}(w).$$

The following diagram might be helpful for understanding the construction of I. (We write $n = \tau(\alpha)$, $n_1 = \tau(\alpha) - 2 - N(\alpha)$, and $n_2 = n_0 + N(\alpha)$.)

$$\begin{tikzcd}
 & \eta^n_{-,n_1}(\alpha) \in \mathbf{\Gamma}_{n_1} \arrow[dr, "\Psi^{n_1}_{+,n-2}"] & & \\
w \in \mathbf{\Gamma}_{n_0} \arrow[ur, "\Psi^{n_0}_{-,n_1}"] \arrow[dr, "\Psi^{n_0}_{+,n_2}"] & & \eta^n_{-,n-2}(\alpha) = \eta^n_{+,n-2}(\alpha) \in \mathbf{\Gamma}_{n-2} & z \in \mathbf{\Gamma}_n \arrow[dl] \arrow[dd, "F_n"] \\
 & \eta^n_{+,n_2}(\alpha) \in \mathbf{\Gamma}_{n_2} \arrow[ur, "\Psi^{n_2}_{-,n-2}"] & \mathbf{\Gamma}_\mu \arrow[l, dotted] \arrow[u, dotted] \arrow[uul, dotted] & \\
I(\alpha) \in \mathbf{Y} \arrow[uu, "G_{n_0}"] & & & \alpha \in \mathbf{Y} \arrow[lll, "I"]
\end{tikzcd}$$

Remark 7.6. For a general 3-manifold Y, our construction of I is noncanonical since there are many choices such as the basis $\mathfrak{B}$ and the element w for each $\alpha \in \mathfrak{B}$. However, one could still ask whether we could simply pick $I = \mathrm{Id}$ or not. If we take $I = \mathrm{Id}$, then Proposition 7.2 can finally be reduced to Conjecture 7.7 which we state below. We believe that the following conjecture is true, though currently we do not find a proof for it. Hence in order to fulfill the main purpose of the paper, we introduce the isomorphism I to bypass this conjecture.

Conjecture 7.7. For any $\alpha \in \mathfrak{B}$, and any integer $n \leqslant \tau(\alpha) - 2$, we have

$$\eta^{\tau(\alpha)}_{\pm,n}(\alpha) = \mathrm{Proj}^{j_\pm}_n \circ G_n(\alpha),$$

where

$$j_\pm = \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2} \mp \frac{(\tau(\alpha) - 2 - n)p}{2}$$

and

$$\mathrm{Proj}^{j_\pm}_n : \mathbf{\Gamma}_n \to (\mathbf{\Gamma}_n, j_\pm)$$

is the projection.

Lemma 7.8. *We have the following.*

(1) *Suppose $\alpha \in \mathfrak{B}$ and n_0, w, $N(\alpha)$ are chosen as above. Suppose n, k are two integers so that $n_0 \leqslant k \leqslant n$. Then (a) $\Psi^k_{-,n} \circ \Psi^{n_0}_{+,k}(w) \neq 0$ if and only if (b) $k \leqslant n_0 + N(\alpha)$ and $n - k \leqslant \tau(\alpha) - 2 - n_0 - N(\alpha)$ (in particular, we have $n \leqslant \tau(\alpha) - 2$).*

(2) *The map $I : \mathbf{Y} \to \mathbf{Y}$ is an isomorphism.*

(3) *For an element $\alpha \in \mathfrak{B}$, an integer n and a grading i, the following two statements are equivalent.*
   (a) *We have $\mathrm{Proj}^i_n \circ G_n \circ I(\alpha) \neq 0$.*
   (b) *We have $n \leqslant \tau(\alpha) - 2$ and there exists $N \in \mathbb{Z}$ so that $N \in [0, \tau(\alpha) - 2 - n]$ and*

$$i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (\tau(\alpha) - 2 - n)p}{2} + Np.$$

(4) *Suppose for an integer* n *and a grading* i *we have* $\alpha_1,\ldots,\alpha_L\in\mathfrak{B}$ *so that* $\mathrm{Proj}_n^i\circ G_n\circ I(\alpha_j)\neq 0$ *for all* $1\leqslant j\leqslant L$, *then* $\mathrm{Proj}_n^i\circ G_n\circ I(\alpha_1),\ldots,\ \mathrm{Proj}_n^i\circ G_n\circ I(\alpha_L)$ *are linearly independent.*

(5) *Suppose* $\alpha\in\mathfrak{B}$. *For any* $n\in\mathbb{Z}$ *so that* $n\leqslant\tau(\alpha)-2$, *we have*

$$\mathrm{Proj}_n^{i_\pm}\circ G_n\circ I(\alpha)=\eta^{\tau(\alpha)}_{\pm,n}(\alpha)\text{ where }i_\pm=\frac{\tau^+(\alpha)+\tau^-(\alpha)\mp(\tau(\alpha)-2-n)}{2}$$

Proof. (1). First, when $k>n_0+N(\alpha)$, from the construction of w, we know that

$$\Psi^{n_0}_{+,k}(w)=\Psi^{n_0+N(\alpha)}_{+,k}\circ\Psi^{n_0}_{+,n_0+N(\alpha)}(w)=\Psi^{n_0+N(\alpha)}_{+,k}\circ\eta^{\tau(\alpha)}_{+,n_0+N(\alpha)}(\alpha)=0.$$

The last equality is from Lemma 6.14 part (6). Similarly, if $n-k>\tau(\alpha)-2-n_0-N(\alpha)$, we know from Lemma 2.11 that

$$\begin{aligned}\Psi^{k}_{-,n}\circ\Psi^{n_0}_{+,k}(w)&=\Psi^{n+n_0-k}_{+,n}\circ\Psi^{n_0}_{-,n+n_0-k}(w)\\&=\Psi^{n+n_0-k}_{+,n}\circ\Psi^{\tau(\alpha)-2-N(\alpha)}_{-,n+n_0-k}\circ\Psi^{n_0}_{-,\tau(\alpha)-2-N(\alpha)}(w)\\\text{(Definition of }w)&=\Psi^{n+n_0-k}_{+,n}\circ\Psi^{\tau(\alpha)-2-N(\alpha)}_{-,n+n_0-k}\circ\eta^{\tau(\alpha)}_{-,\tau(\alpha)-2-N(\alpha)}(\alpha)\\\text{(Lemma 6.14 part (6))}&=0.\end{aligned}$$

Next, we need to show that $\Psi^{k}_{-,n}\circ\Psi^{n_0}_{+,k}(w)\neq0$ when $k\leqslant n_0+N(\alpha)$ and $n-k\leqslant\tau(\alpha)-2-n_0-N(\alpha)$. Again, from Lemma 2.11 we have

$$\begin{aligned}\Psi^{n}_{+,n+N(\alpha)+n_0-k}\circ\Psi^{k}_{-,n}\circ\Psi^{n_0}_{+,k}(w)&=\Psi^{n_0+N(\alpha)}_{-,n_0+N(\alpha)+n-k}\circ\Psi^{n_0}_{+,n_0+N(\alpha)}(w)\\\text{(Definition of }w)&=\Psi^{n_0+N(\alpha)}_{-,n_0+N(\alpha)+n-k}\circ\eta^{\tau(\alpha)}_{+,n_0+N(\alpha)}(\alpha)\\\text{(Lemma 6.14 part (6))}&=\eta^{\tau(\alpha)}_{+,n_0+N(\alpha)+n-k}(\alpha)\\\text{(Lemma 6.14 part (3) and (6))}&\neq0.\end{aligned}$$

(2). Suppose $\mathfrak{B}=\{\alpha_1,\ldots,\alpha_L\}$ where $L=\dim_{\mathbb{C}}\mathbf{Y}$. We order the elements α_i so that

$$\tau(\alpha_i)\geqslant\tau(\alpha_{i+1}).$$

Let w_i, $N_i=N(\alpha_i)$ be the data associated to α_i as above. Since

$$w_i\in\bigoplus_{j\in(-\frac{p}{2},\frac{p}{2}]}(\mathbf{\Gamma}_{n_0},j)$$

for any i, and by Lemma 2.19, the map

$$\mathrm{Proj}\circ G_{n_0}:\mathbf{Y}\to\bigoplus_{j\in(-\frac{p}{2},\frac{p}{2}]}(\mathbf{\Gamma}_{n_0},j)$$

is an isomorphism, in order to show that I is an isomorphism, it suffices to show that $w_1,\ldots,w_L$ are linearly independent.

Now suppose there are complex numbers $\lambda_1,\ldots,\lambda_L$ so that

$$\sum_{i=1}^{L}\lambda_iw_i=0.$$

Applying the map $\Psi^{n_0+N_1}_{+,\tau(\alpha_1)-2}\circ\Psi^{n_0}_{-,n_0+N_1}$, from the construction of w_i, the order of α_i and part (1) of the lemma, we know

$$\begin{aligned}0&=\Psi^{n_0+N_1}_{+,\tau(\alpha_1)-2}\circ\Psi^{n_0}_{-,n_0+N_1}(\sum_{i=1}^{L}\lambda_i w_i)\\&=\sum_{\alpha_i}\lambda_i\alpha_i\end{aligned}$$

where the summation in the second line is over all α_i with

$$\tau(\alpha_i)=\tau(\alpha_1),\text{ and }N_i=N_1.$$

Since α_i are linearly independent, so all relevant λ_i must be zero. Suppose i_0 is the smallest index in the rest. By our choice of α_i, the element α_{i_0} has the largest τ among the rest α_i. Hence we can apply the map $\Psi^{n_0+N_{i_0}}_{+,\tau(\alpha_{i_0})-2}\circ\Psi^{n_0}_{-,n_0+N_{i_0}}$ to filter out α_i with smaller τ. Repeating this argument, we could prove that all λ_i must be zero.

(3). Let n_0, w, $i(\alpha)$, and $N(\alpha)$ be constructed as above. We first prove that (b) $\Rightarrow$ (a). Note when constructing the isomorphism I, from Corollary 2.9 and Lemma 2.18 we can take $n_0'=n_0-2$ and $w'=(\psi^{n_0-2}_{+,n_0-1})^{-1}\circ(\psi^{n_0-1}_{+,n_0})^{-1}(w)$ that will lead to the same I as n_0, w. (Note by construction passing from n_0 to n_0-2 will increase $N(\alpha)$ by 1.) As a result, we can always assume that n_0 is small enough compared with any given n. Now recall by construction

$$i(\alpha)=\frac{\tau^+(\alpha)+\tau^-(\alpha)}{2}-\frac{(\tau(\alpha)-2-n_0)p}{2}+N(\alpha)p$$

and by the assumption in (b) we have

$$i=\frac{\tau^+(\alpha)+\tau^-(\alpha)-(\tau(\alpha)-2-n)p}{2}+Np.$$

We can assume that n_0 is small enough so that $N(\alpha)>N$. Take $k=N(\alpha)-N+n_0$. It is straightforward to check that from the construction of I and Lemma 2.18 we have

$$\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)=\Psi^k_{-,n}\circ\Psi^{n_0}_{+,k}(w).$$

It is straightforward to verify that $k-n_0\leqslant N(\alpha)$ and $n-k\leqslant\tau(\alpha)-2-n_0-N(\alpha)$. As a result, we conclude from part (1) that

$$\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)\neq 0.$$

Next we show that (a) $\Rightarrow$ (b). Again assume that n_0 is small enough compared with the given n. Then there exists $i'\in(-\frac{p}{2},\frac{p}{2}]$ so that there exists $N'\in\mathbb{Z}$ with

$$i'=i-\frac{(n-n_0)p}{2}+N'p.$$

By Lemma 2.18, we know that

$$\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)=\Psi^{n_0+N'}_{-,n}\circ\Psi^{n_0}_{+,n_0+N'}\circ\mathrm{Proj}^{i'}_{n_0}\circ G_{n_0}\circ I(\alpha).$$

From the construction of $I(\alpha)$ and Lemma 2.19 we know $\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)\neq 0$ only if $i'=i(\alpha)$, in which case

$$\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)=\Psi^{n_0+N'}_{-,n}\circ\Psi^{n_0}_{+,n_0+N'}\circ\mathrm{Proj}^{i'}_{n_0}\circ G_{n_0}\circ I(\alpha)=\Psi^{n_0+N'}_{-,n}\circ\Psi^{n_0}_{+,n_0+N'}(w).$$

Hence $\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)\neq 0$ implies that

$$N'\leqslant N(\alpha)\text{ and }n-N'\leqslant\tau(\alpha)-2-N(\alpha)$$

by part (1). Taking $N = N(\alpha) - N'$, it is then straightforward to check that

$$N \in [0, \tau(\alpha) - 2 - n] \text{ and } i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (\tau(\alpha) - 2 - n)p}{2} + Np.$$

(4). The proof is similar to that of (2).

(5). It follows from the proofs of part (1) and (3). □

7.3. **The exact triangle.** In this subsection, we prove the exact triangle. Note that we choose the basis $\mathfrak{B}$ of $\mathbf{Y}$ as in Section 6.3.

Proof of Proposition 7.1. We will verify the exactness at each space of the triangle.

The exactness at $\mathbf{\Gamma}_{n+k} \oplus \mathbf{\Gamma}_{n+k}$. This follows from Proposition 5.1.

The exactness at $\mathbf{\Gamma}_n$. From Lemma 7.3 and the construction of Φ_n^{n+2k} in (7.2), we know that $\operatorname{Im}\Phi_n^{n+2k} \subset \ker \Phi_{n+k}^n$. Now pick an arbitrary

$$x \in (\mathbf{\Gamma}_n, i) \cap \ker \Phi_{n+k}^n = \operatorname{Im}\left(\operatorname{Proj}_n^i \circ G_n\right)$$

Since I is an isomorphism, we can assume that

$$x = \sum_{j=1}^{l} \operatorname{Proj}_n^i \circ G_n(\lambda_j \cdot I(\alpha_j))$$

where $\alpha_j \in \mathfrak{B}$ and $\operatorname{Proj}_n^i \circ G_n \circ I(\alpha_j) \neq 0$. From Lemma 7.8 part (3), we know that this implies that for any $j \in [1, l] \cap \mathbb{Z}$, we have $n \leqslant \tau(\alpha_j) - 2$ and there exists $N_j \in \mathbb{Z}$ so that $N_j \in [0, \tau(\alpha_j) - 2 - n]$

$$i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (\tau(\alpha) - 2 - n)p}{2} + N_j p.$$

Now for k large enough, we have $n + 2k > \tau(\alpha_j)$. Taking

$$N_j' = n + k + 1 - \tau(\alpha) + N_j \in \mathbb{Z},$$

it is straightforward to verify that when k is large enough, we have

$$N_j' \in [0, n + 2k - \tau(\alpha)] \text{ and } \ i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (n + 2k - \tau(\alpha))p}{2} + N_j' p.$$

Hence by Lemma 6.12 part (3), there exists $y_j \in (\mathbf{\Gamma}_{n+2k}, i)$ so that $F_{n+2k}(y_j) = \alpha_j$. As a result, it is straightforward to check that

$$x = \Phi_n^{n+2k}\left(\sum_{i=1}^{l} \lambda_i \cdot y_i\right) \in \operatorname{Im}\Phi_n^{n+2k}.$$

The exactness at $\mathbf{\Gamma}_{n+2k}$. Suppose $x \in (\mathbf{\Gamma}_{n+2k}, i)$ and

$$F_{n+2k}(x) = \sum_{j=1}^{l} \lambda_j \cdot \alpha_j$$

with $\lambda_j \neq 0$ and $\alpha_j \in \mathfrak{B}$.

First, if $x \in \operatorname{Im}\Phi_{n+2k}^{n+k}$, then from Lemma 7.4 part (2), we know that for any $1 \leqslant j \leqslant l$, we have

$$\text{either } i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p - q}{2} \text{ or } i \leqslant \tau^+(\alpha_j) + \frac{(n-1)p - q}{2}.$$

If we write

$$i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (\tau(\alpha) - 2 - n)p}{2} + N_j p,$$

for some N_j then the inequality

$$i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p - q}{2}$$

implies that

$$\begin{aligned} N_j &\geqslant \frac{1}{p}\left(\tau^-(\alpha_j) - \frac{(n-1)p-q}{2} - \frac{\tau^+(\alpha) + \tau^-(\alpha) - (\tau(\alpha) - 2 - n)p}{2}\right) \\ &= \frac{1}{2}\left(1 + \frac{\tau^-(\alpha_j) - \tau^+(\alpha_j) + q}{p}\right) + \frac{\tau(\alpha)}{2} - 1 - n \\ &= \tau(\alpha) - 1 - n. \end{aligned}$$

Note the last equality uses the definition of $\tau(\alpha)$ in Definition 6.8. Similarly we can compute that

$$i \leqslant \tau^+(\alpha_j) + \frac{(n-1)p - q}{2}$$

implies that

$$\begin{aligned} N_j &\leqslant \frac{1}{p}\left(\tau^+(\alpha_j) + \frac{(n-1)p-q}{2} - \frac{\tau^+(\alpha) + \tau^-(\alpha) - (\tau(\alpha) - 2 - n)p}{2}\right) \\ &= \frac{1}{2}\left(-1 + \frac{\tau^+(\alpha_j) - \tau^-(\alpha_j) - q}{p}\right) + \frac{\tau(\alpha)}{2} - 1 \\ &= -1. \end{aligned}$$

In summary, $x \in \operatorname{Im} \Phi^{n+k}_{n+2k}$ implies that for all $1 \leqslant j \leqslant l$, either $N_j \geqslant \tau(\alpha_j) - 1 - n$ or $N_j \leqslant -1$. Hence from Lemma 7.8 part (3), we know that

$$\operatorname{Proj}^i_n \circ G_n \circ I(\alpha_j) = 0$$

for all $1 \leqslant j \leqslant l$ and as a result, $\Phi^{n+2k}_n(x) = 0$.

Second, suppose $x \notin \operatorname{Im} \Phi^{n+k}_{n+2k}$. For any $1 \leqslant j \leqslant l$, we can write

$$i = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (\tau(\alpha) - 2 - n)p}{2} + N_j p$$

for some N_j. Then from Lemma 7.4 part (3) and a similar computation as above, we know that there exists j so that $1 \leqslant j \leqslant l$, and

$$N_j \in [0, \tau(\alpha_j) - 2 - n] \cap \mathbb{Z}.$$

Hence by Lemma 7.8 part (2) and (4) we know that

$$\operatorname{Proj}^i_n \circ G_n \circ I(\alpha_j) \neq 0 \Rightarrow \Phi^{n+2k}_n(x) \neq 0.$$

Hence we conclude that

$$\operatorname{Im} \Phi^{n+k}_{n+2k} = \ker \Phi^{n+2k}_n.$$

□

7.4. **The commutative diagram.** In this subsection, we will prove the commutative diagram presented at the beginning of the section. Note that we choose the basis $\mathfrak{B}$ of $\mathbf{Y}$ as in Section 6.3.

Lemma 7.9. *Suppose $n \in \mathbb{Z}$ and i is a grading. Suppose $x \in (\mathbf{\Gamma}_n, i)$ so that*

$$F_n(x) = \sum_{j}^{l} \lambda_j \alpha_j,$$

with $\lambda_i \neq 0$ and $\alpha_j \in \mathfrak{B}$ for all $1 \leqslant j \leqslant l$. Then for any $1 \leqslant j \leqslant l$, there exists $N_j \in [0, n+1-\tau(\alpha_j)]$ so that

$$i = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (n+1-\tau(\alpha_j))p}{2} + N_j p.$$

Proof. This is a combination of Lemma 6.11 part (3), Lemma 6.12 part (3), and Lemma 7.4 part (1). The proof is similar to that of Lemma 7.4 part (2). ☐

Proof of Proposition 7.2. We only prove the first commutative diagram

$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n+2k+1}{2}} \arrow[r, "\psi^{n+k+1}_{+,\mu} \circ \psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"] \arrow[d, "\Psi^{n+k+1}_{+,n+2k} \circ \psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"'] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^{\mu}_{+,n}"] \\
\mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_n"] & \mathbf{\Gamma}_n
\end{tikzcd}$$

The other is similar. Note that at the end of Section 6.3, we introduce new notations of $\eta^n_{\pm,n-2}$ to remove the scalars. Then the second commutative diagram only holds up to a scalar.

First, note the maps from $\mathbf{\Gamma}_{\frac{2n+2k+1}{2}}$ to $\mathbf{\Gamma}_\mu$ and $\mathbf{\Gamma}_{n+2k}$ both factor through $\mathbf{\Gamma}_{n+k+1}$. As a result, we only need to prove the following commutative diagram for large enough k.

$$\begin{tikzcd}
\mathbf{\Gamma}_{n+k+1} \arrow[r, "\psi^{n+k+1}_{+,\mu}"] \arrow[d, "\Psi^{n+k+1}_{+,n+2k}"'] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^{\mu}_{+,n}"] \\
\mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_n"] & \mathbf{\Gamma}_n
\end{tikzcd}$$

Now suppose $x \in (\mathbf{\Gamma}_{n+k+1}, i)$. Write

$$F_{n+k+1}(x) = \sum_{j=1}^{l} \lambda_i \cdot \alpha_j$$

with $\lambda_j \neq 0$ and $\alpha_j \in \mathfrak{B}$ for $1 \leqslant j \leqslant l$. From Lemma 7.9, we know for any $1 \leqslant j \leqslant l$, there exists $N_j \in [0, n+k+1-\tau(\alpha_j)]$ so that

$$i = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (n+k+1-\tau(\alpha_j))p}{2} + N_j p. \tag{7.4}$$

Taking $n'_j = \tau(\alpha_j)$ and $N'_j = 0$, we can apply Lemma 6.12 part (3) to find an element

$$x_j \in (\mathbf{\Gamma}_{\tau(\alpha_j)}, \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j)}{2})$$

so that

$$F_{\tau(\alpha_j)}(x_j) = \alpha_j.$$

It is then straightforward to check that

$$y_j = \Psi^{\tau(\alpha_j)+N_j}_{+,n+k+1} \circ \Psi^{\tau(\alpha_j)}_{-,\tau(\alpha_j)+N_j}(x_j) \in (\mathbf{\Gamma}_{n+k+1}, i). \tag{7.5}$$

Write

$$y = x - \sum_{i=1}^{l} \lambda_i \cdot y_j \in (\mathbf{\Gamma}_{n+k+1}, i).$$

From Lemma 2.18 we know that

$$F_{n+k+1}(y) = 0.$$

As a result, by Lemma 6.6,

$$\psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(x) = \sum_{i=1}^{l} \lambda_i \cdot \psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(y_j)$$

Note unless $N_j = n + k + 1 - \tau(\alpha_j)$, we have

$$\psi^{n+k+1}_{+,\mu} \circ \Psi^{\tau(\alpha_j)+N_j}_{+,n+k+1} = 0$$

by the exactness. As a result,

$$\begin{aligned}
\psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(x) &= \sum_{\substack{1\leqslant j\leqslant l \\ N_j = n+k+1-\tau(\alpha_j)}} \lambda_i \cdot \psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu} \circ \Psi^{\tau(\alpha_j)}_{-,\tau(\alpha_j)+N_j}(x_j) \\
\text{(Commutativity in Lemma 2.12)} &= \sum_{\substack{1\leqslant j\leqslant l \\ N_j = n+k+1-\tau(\alpha_j)}} \lambda_i \cdot \psi^{\mu}_{+,n} \circ \psi^{\tau(\alpha_j)}_{+,\mu}(x_j) \\
\text{(Definition of } \eta^{\tau(\alpha_j)}_{+,n}) &= \sum_{\substack{1\leqslant j\leqslant l \\ N_j = n+k+1-\tau(\alpha_j)}} \lambda_i \cdot \eta^{\tau(\alpha_j)}_{+,n}(\alpha_j)
\end{aligned}$$

Now we deal with $\Phi^{n+2k}_n \circ \Psi^{n+k+1}_{+,n+2k}(x)$. Since $F_{n+k+1}(y) = 0$, Lemma 2.19 implies that

$$\Psi^{n+k+1}_{+,n+2k}(y) = 0.$$

Hence

$$\Psi^{n+k+1}_{+,n+2k}(x) = \sum_{j=1}^{l} \lambda_i \cdot \Psi^{n+k+1}_{+,n+2k}(y_j)$$

where y_j is defined as in (7.5). Note by definition $y_j \in (\mathbf{\Gamma}_{n+1+k}, i)$, so from Lemma 2.6, we know

$$\Psi^{n+k+1}_{+,n+2k}(y_j) \in (\mathbf{\Gamma}_{n+2k}, i - \frac{(k-1)p}{2}).$$

Note by (7.5) and Lemma 2.18, we know that

$$F_{n+2k} \circ \Psi^{n+k+1}_{+,n+2k}(y_j) = F_{\tau(\alpha_j)}(x_j) = \alpha_j.$$

Hence

$$\Phi_n^{n+2k}\circ\Psi_{+,n+2k}^{n+k+1}(x)=\sum_{i=1}^{l}\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}}\circ G_n\circ I(\alpha_j).$$

We write

$$i-\frac{(k-1)q}{2}=\frac{\tau^+(\alpha_j)+\tau^-(\alpha_j)-(\tau(\alpha_j)-2-n)p}{2}+N_j'p$$

Comparing the above formula with (7.4), we know

$$N_j'=N_j+\tau(\alpha_j)-n-k-1.$$

Note by construction $N_j\leqslant n+k+1$ which means $N_j'\leqslant 0$. Hence from Lemma 7.8 we know

$$\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}}\circ G_n\circ I(\alpha_j)\neq 0$$

if and only if $N_j'=0$, *i.e.*, $N_j=n+k+1-\tau(\alpha)$. Also when $N_j'=0$ from Lemma 7.8 part (5) we know

$$\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}}\circ G_n\circ I(\alpha_j)=\eta_{+,n}^{\tau(\alpha_j)}(\alpha_j).$$

As a result, we know

$$\begin{aligned}\Phi_n^{n+2k}\circ\Psi_{+,n+2k}^{n+k+1}(x)&=\sum_{j=1}^{l}\lambda_j\cdot\mathrm{Proj}_n^{j-\frac{(k-1)p}{2}}\circ G_n\circ I(\alpha_j)\\&=\sum_{\substack{1\leqslant j\leqslant l\\ N_j=n+k+1-\tau(\alpha_j)}}\lambda_j\cdot\eta_{+,n}^{\tau(\alpha_j)}(\alpha_j)\\&=\psi_{+,n}^{\mu}\circ\psi_{+,\mu}^{n+k+1}(x)\end{aligned}$$

□

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